Fast and Efficient Restricted Delaunay Triangulation in Random Geometric Graphs

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Abstract. Let $G = \mathcal{G}(n, r)$ be a random geometric graph resulting from placing n nodes uniformly at random in the unit square (disk) and connecting every two nodes if and only if their Euclidean distance is at most r . Let $r_{\text{con}} = \sqrt{\frac{\log n}{\pi n}}$ be the known critical radius guaranteeing connectivity when $n \to \infty$. The *Restricted Delaunay Graph* $RDG(G)$ is a subgraph of G with the following properties: it is a planar graph and a spanner of G, and in particular it contains all the short edges of the Delaunay triangulation of G. While in general networks the construction of $RDG(G)$ requires $O(n)$ messages, we show that when $r = O(r_{\text{con}})$ and $G = \mathcal{G}(n, r)$, with high probability, $RDG(G)$ can be constructed locally in one round of communication with $O(\sqrt{n \log n})$ messages, and with only one hop neighborhood information. This proves that the existence of long Delaunay edges (an order larger than r_{con}) in the unit square (disk) does not significantly impact the efficiency with which good routing graphs can be maintained.

1 Introduction

A random geometric graph is a graph $\mathcal{G}(n, r)$ resulting from placing n points uniformly at random in the unit square and connecting two points if and only if their Euclidean distance is at most r. These random graphs have long been the subject of studies with relation to topics such as statistical physics and hypothesis testing [23]. They have since gained new relevance as a model of random wireless networks, in large part due to the advance in the field of sensor networks [12, 24]. Sensor networks are constructed from a large number of low cost, low power sensors equipped with wireless communication and limited processing capabilities. These devices are expected to be embedded densely into the environment to create a multi hop network in which nodes can cooperate to achieve high level tasks, and a wide range of applications have been offered in the last few years, ranging from environmental and habitat monitoring to disaster management and manufacturing process flow [9].

Sensor Networks introduce new design challenges; the strict energy and memory constraints of the sensors and the large scale of the network require the use of distributed, localized algorithms which minimize memory and energy use [12].

Since energy is mostly consumed by radio communication, the number of messages being sent by a given algorithm is considered the efficiency metric. Naturally, these restrictions and the theoretical model of random geometric graphs have led to a variety of analytical work aimed at investigating different properties of such networks [16, 15, 21, 2].

Among others, the tasks of topological control and routing in these networks have been studied extensivley, and in particular geographical routing or *geo-routing* approaches have been established [6, 17]. In geo-routing the assumption is that each node knows its own location (i.e. coordinates) and the location of the destination to which it wants to deliver a message (via Global Positioning System (GPS) at each node or some other mechanism). The goal then is to find an efficient route from source to destination using only local information (available at each node) and limited memory. Most of the early work on this issue, beginning with the proposals of Bose *et al* [6] and Karp and Kung [17], has been based on greedy forwarding combined with *face routing* over a *planar* subgraph of the network. That is, the message is always forwarded to the neighbor closest to the destination, and if such a neighbor does not exist, recovery from the local minima is obtained using a route along the current face of the planar graph. Although this method guarantees delivery, the efficiency of the method depends on the properties of the planar subgraph. Ideally, our subgraph should be sparse and locally constructed, but at the same time a spanner. The sparseness and locality reduce energy and memory consumption while the spanner property shows that a short route (compared to the original graph) does exist in the subgraph.

Several such structures have been offered recently in the literature. The *Relative Neighborhood graph*, $RNG(G)$, [25] and the *Gabriel Graph*, $GG(G)$, [13] are both planar and can be efficiently constructed locally, but are not good spanners, even in random graphs [4]. Another well-known planar graph of G is the Delaunay triangulation, $Del(G)$. It is also a spanner of the complete graph [7, 11], but it cannot be constructed locally and may contain long edges. In other words $Del(G)$ is not necessarily a subgraph of G. To overcome this problem several authors proposed the *Restricted Delaunay Graph*, $RDG(G)$. This is a planar graph that contains all the edges of $Del(G)$ that are also in G, and as proved by [14] and [19] it is also a spanner of G. Note, that by definition $RDG(G)$ is not unique and different methods have been suggested to construct such graphs [14, 19, 26, 1].

In the context of random geometric graphs, we are interested in the relation between the range of communication r, the number of nodes n in the graph and some desired property P (for example connectivity). In ad-hoc and sensor networks interference grows with increased communication radius. So one wants to find a tight upper bound on the smallest radius r_P, that will guarantee that P holds with high probability[1]. For example, the critical radius for connectivity, r_{con}, has been of special interest, and it has been shown that if $\pi r^2 \geq \pi r_{\text{con}}^2 =$

[1] Event $\mathcal{E}_n$ occurs with high probability if probability $P(\mathcal{E}_n)$ is such that $\lim_{n\to\infty} P(\mathcal{E}_n) = 1$.

$\frac{\log n + \gamma_n}{n}$ then $\mathcal{G}(n, r)$ is connected with probability going to one as $n \to +\infty$ iff $\gamma_n \to +\infty$ [22, 16].

It is well known that the maximum edge length of the Delaunay triangulation of $\mathcal{G}(n, r)$ in the unit square, and in particular on the convex hull, is $w(r_{\text{con}})$. Recently, a similar result has been proved also for the unit disk [18]. Therefore it is clear that when $r = O(r_{\text{con}})$ the Delaunay triangulation cannot be computed locally (i.e. with information obtained only from nodes that are a constant number of hops away).

In this paper we show that if $r = O(r_{\text{con}})$, namely on the order that guarantees connectivity, then with high probability, we can efficiently and locally construct a *Restricted Delaunay Graph* $RDG(G)$. While for general graphs this had been accomplished with $\Theta(n)$ messages, we show that only an order of $O(\sqrt{n \log n})$ messages is needed in $\mathcal{G}(n, r)$ and present a novel algorithm that achieves this bound. Two properties differentiate our algorithm and allows it to reduce the total number of messages. First, it requires only one round of communication and second, only problematic nodes send messages. Our results are stated for geometric graphs that have some nice properties, but are not necessarily random nor in a specific bounded area (i.e. square, disk). Later we show that *random geometric graphs* in the unit square (or unit disk) have these nice properties with high probability and the results follow.

2 Preliminaries

We consider a wireless ad hoc network (or sensor network) over a set V of n nodes distributed in the unit square, where each node can communicate with all the nodes in its transmission range, that is a disk of radius r centered at the node. The resulting is a *geometric graph* $G = G(V, r)$ with V the set of nodes and $E = \{(u, v) \mid u, v \in V \wedge ||uv|| \leq r\}$ the set of edges. This is similar to the *Unit Disk Graph* $UDG(V)$ [8] in which the set of nodes $V \in \mathbb{R}^2$ and the radius is assumed to be one unit, but in our case we are interested in a network in a bounded area and in the relation between the number of nodes n and the transmission range r as a function of n.

Let $N(u)$ denote the neighbors of u including u and $N(u, v)$ the set of the common neighbors of u and v, i.e. $N(u, v) = N(v, u) = N(u) \cap N(v)$. Throughout the paper we use three disk definitions: let $disk_r(v)$ be the disk centered around v with radius r (with r omitted when the context is clear), $disk(u, v)$ be the disk through u, v with diameter $||uv||$ and $disk(u, v, w)$ be the unique circumcircle over u, v and w.

Next we present more graphs over the set of nodes V, note that in some cases the graphs are derived directly from V, others are a function of G (i.e. r is needed to compute them). The *Voronoi* diagram of a set of nodes (or sites) V in the space, $Vor(V)$, is the partition of the space into cells C_u, $u \in V$ such that all the points inside C_u are closer to u than to any other node in V. The *Delaunay triangulation*, $Del(V)$, is the dual graph of *Vor(V)*: an edge (u, v) is in $Del(V)$ if and only if C_u and C_v share a common boundary. It is well known that $Del(V)$ is a spanner of the complete graph K_n [7, 11], which means that the shortest path between any two points on $Del(V)$ is at most t times the shortest path on K_n, where t is a positive constant called the *stretch factor*. A useful property of the Delaunay triangulation is that a triangle $\triangle uvw \in Del(V)$ if and only if $disk(u, v, w)$ is empty: there is no other node from V in it[2] [10].

Let $UDel(G)$ be the subgraph of $Del(V)$ that contains only the short edges of $Del(V)$, that is the edges that are shorter than r; therefore? $UDel(G) = Del(V) \cap G$ and is also a subgraph of G [19, 14]. By a *Restricted Delaunay Graph* $RDG(G)$ we will mean a planar graph such that $UDel(G) \subseteq RDG(G) \subseteq G$ [14].

Let $T(u)$ be the set of edges in $Del(N(u))$ (i.e the Delaunay triangulation of the nodes in $N(u)$) and similarly $T(u, v) = Del(N(u, v))$. Note that there may be edges in $T(u)$ and $T(u, v)$ which are not presented in $Del(G)$. The graph $LocalDel(G)$ is the graph resulting from computing $T(u)$ at each node: an edge (u, v) is in $LocalDel(G)$ if and only if there exist u s.t $(u, v) \in T(u)$. Figure 1 illustrates the graphs discussed above for a set V of 50 random points in the unit square.

3 Related Work

Unit Disk Graphs: The *Gabriel Graph*, $GG(G)$, [13] is a graph where there is an edge (u, v) if and only if there is no other node in $disk(u, v)$. Bose *et al* [6] offered a distributed local algorithm to construct the Gabriel Graph over a wireless network and then used face routing to guarantee message delivery. Later, Bose and Morin [5] considered different face routing methods in triangulation and in particular in the Delaunay triangulation. In [17], Karp and Kung independently proposed GPSR, a memory-less routing algorithm that combines greedy forwarding and local minimum recovery that is also based on face routing over the Gabriel Graph. Subsequently, later work aimed at finding better planar graphs that can be constructed locally.

Gao *et al* [14] proposed the use of a Restricted Delaunay Graph $RDG(G)$, a graph that contains all the short edges of the Delaunay Graph and is also planar. The authors proved that $RDG(G)$ is Euclidean spanner of the Unit Disk Graph G and gave an algorithm to construct it that, in general, can be inefficient with $O(n^2)$ messages. Similarly, Li *et al* [19] proved that the $UDel(G)$ is a spanner of the Unit Disk Graph G and offered a local algorithm to build a planar supergraph of $UDel(G)$[3] called $PLDel(G)$ in $\Theta(n)$ messages and $\Theta(n \log n)$ bits. They presented yet another graph, $LDel^{(k)}(G)$, a local Delaunay triangulation where the circumcircle of u, v, w does not contain any node which is k hops away from u, v or w. The authors proved that $LDel^{(k)}(G)$, $k \geq 1$ is a supergraph of $UDel(G)$ and a subgraph of $LDel^{(k+1)}(G)$ and therefore a spanner. In addition they showed that for $k = 1$ $LDel^{(k)}(G)$ is not planar, but for $k > 1$ it is. Recently Wang and Li [26] showed how to bound the maximum degree of such graphs, since $PLDel(G)$ or in general $UDel(G)$ are not bounded degree graphs.

[2] For simplicity we assume that no four points in V are co-circular

[3] $RDG(G)$ in [14] notation.

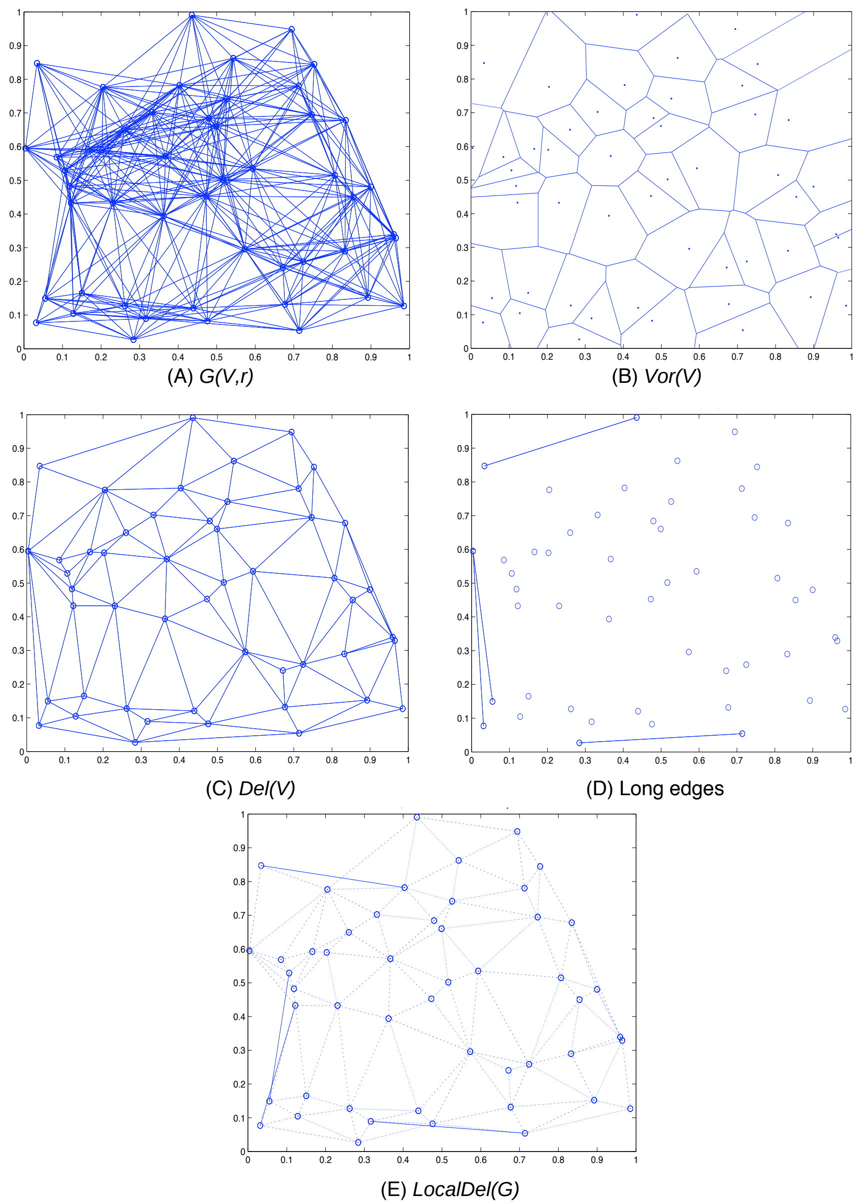

Fig. 1. Different Graphs over a set V of 50 random nodes in the unit square with $r = 0.3$ (A) $G(V, r)$. (B) $Vor(V)$. (C) $Del(V)$. (D) The edges in $Del(V)$ that are longer than r (E) $LocalDel(G)$ where consistent edges are in dots and inconsistent edges are in solid lines.

Arajo and Rodrigues [1] reduced the number of steps in [19] but their algorithm still has the same order of messages, $\Theta(n)$. All of the above algorithms are non adaptive: in some cases they send unnecessary messages. Essentially, they require each node u to broadcast all the triangles in $T(u)$ with a $\measuredangle wux \geq \pi/3$. Since the total number of such triangles (faces) in the above graphs is linear, all the algorithms require a linear number of messages.

Random Geometric Graphs: Bose *et al* [4] proved (among other results) that the Gabriel Graph is not a spanner of the Unit Disk Graph and that, in the worst case, its stretch factor is $\Theta(\sqrt{n})$. Moreover, they proved that for random geometric graphs in the unit square the stretch factor of the Gabriel Graph is with high probability $\Theta(\sqrt{\log n/\log\log n})$, which proves its inefficiency for face routing in random networks. In [18], Kozma *et al* bounded the longest edge of $Del(G)$ in a random geometric graph in the unit disk. They showed that, due to boundary effects, the longest edge is of order $O(\sqrt[3]{\frac{\log n}{n}})$, an order larger than r_{con}, and left open the question of an algorithm for the case where $r = O(r_{\mathrm{con}})$. Bern *et al* [3] proved that the Delaunay triangulation of a uniform set of points does not have bounded degree, and that the maximum degree grows like $\Theta(\log n/\log\log n)$. In particular, they showed that this does not happen next to the boundary. Since higher degree leads to greater load imbalance, one wants to have a constant degree planar graph; in our case, the algorithm we offer does not solve this problem.

4 Computing $RDG(\mathcal{G}(n,r))$

We offer an efficient algorithm to construct $RDG(\mathcal{G}(n,r))$. There are several advantage to our algorithm: (i) it assumes the transmission range is on the same order as the necessary range required for connectivity. (ii) there is only one round of communication, and most importantly (iii) the number of messages it sends it adaptive. Our algorithm is based on $LocalDel(G)$ which is known not to be a planar graph. There has been previous proposal to solve this problem but all of them required a constant number of messages per node. In our algorithm only messages that are needed to eliminate problematic edges are sent, enabling us to reduce the number of messages from $O(n)$ to $O(\sqrt{n\log n})$. Before describing the algorithm we define the following:

Definition 1 *A* ***local inconsistent*** *edge at u in an edge* (u,v) *s.t* $(u,v) \notin T(u)$ *and* $(u,v) \in T(u,v)$ *and a* ***local consistent*** *otherwise.*

Claim 1 *For each local inconsistent edge* $(u,v) \in T(u,v)$ *there is a triangle* $\triangle uwx \in T(u)$ *which is the* ***proof*** *that* (u,v) *is local inconsistent, i.e.* (w,x) *intersect with* (u,v)*.*

Now we can introduce our algorithm:

The main results of this work are the following theorems about the correctness and number of messages in Algorithm 1:

Algorithm 1. **local RDG(G) construction at node u**

1 Compute $T(u)$ and for each neighbor $v \in N(u)$ compute $T(u,v)$.
2 Keep all edges $(u,v) \in T(u)$.
3 If there are local inconsistent edges, broadcast *proofs* for all of them.
4 Remove edge (u,v) if received a *proof* of its inconsistency.

Theorem 1 *For $r \geq \sqrt{\frac{32}{3}} r_{\mathrm{con}}$ Algorithm 1 computes $RDG(\mathcal{G}(n,r))$ w.h.p.*

Theorem 2 *For $r \geq \sqrt{\frac{32}{3}} r_{\mathrm{con}}$ the number of messages in Algorithm 1 is $O(\sqrt{n \log n})$ and the number of bits is $O(\sqrt{n}(\log n)^{3/2})$ w.h.p.*

To prove this theorems we next establish a few helping lemmas.

5 Properties of *LocalDel(G)*

The $LocalDel(G)$ graph can be constructed locally without exchanging messages assuming that each node knows the locations of all its neighbors. We assume that this information is obtained by each node using some other mechanism that is shared with other applications. We therefore do not count the messages required in this process as part of our algorithm (otherwise $\Omega(n)$ messages are necessary for any task) and only consider algorithm-specific messages. Next we state some properties of $LocalDel(G)$ that are based on the following proposition:

Proposition 1 *Let $V' \subseteq V$. $(u, v \in V' \wedge (u,v) \in Del(V)) \Rightarrow (u,v) \in Del(V')$*

This clearly follows from the fact that if two Voronoi cells C_u and C_v share a boundary in $Vor(V)$ they must share a boundary in $Vor(V')$, since removing nodes cannot decrease their boundary.

Definition 2 *An edge (u,v) in $LocalDel(G)$ is* ***consistent*** *if $(u,v) \in T(u)$ and $(u,v) \in T(v)$, and* ***inconsistent*** *otherwise.*

Lemma 2 *If $(u,v) \in UDel(G)$, then (u,v) is a consistent edge in LocalDel(G).*

Proof: This follows directly from Proposition 1. Since $(u,v) \in UDel(V)$ we have $(u,v) \in Del(V)$, $u \in N(v)$ and $v \in N(u)$, so with $V' = N(u,v)$ we get $(u,v) \in T(u) = Del(V')$ and $(v,u) \in T(v) = Del(V')$ □

It is clear from Lemma 2 that $UDel(G) \subseteq LocalDel(G)$, however, it is still not $RDG(G)$ since it may be not a planar. There are two type of edges in $LocalDel(G)$: consistent and inconsistent, and both may cross other edges. First we take care of the inconsistent edges.

Lemma 3 *An edge* $(u, v) \in LocalDel(G)$ *is* inconsistent *if and only if* (u, v) *is* local inconsistent *at* u *or* v*.*

Proof: $\Rightarrow$: Assume edge $(u, v) \in LocalDel(G)$ is inconsistent, w.l.o.g let $(u, v) \in T(u)$ and $(v, u) \notin T(v)$. By Proposition 1 $(u, v) \in T(u, v) = T(v, u)$ so (v, u) must be local inconsistent at v. $\Leftarrow$: By Proposition 1 if an edge is consistent it must be local consistent at u and v. □

Next we bound the number of *proofs* each node can have for its inconsistent edges

Lemma 4 *A node can have at most 6* proofs *for all its local inconsistent edges in* $LocalDel(G)$*.*

Proof: A triangle $\triangle uwx \in T(u)$ with $\measuredangle wux \leq \pi/3$ cannot be a proof for a local inconsistent edge (u, v) since v must then be neighbor with w and x and $Del(N(u))$ and $Del(N(u, v))$ agree on (u, v) and (w, x). □

5.1 Well-distributed Geometric Graphs

From now on we turn to a more specific type of geometric graphs. First lets define them formally:

Definition 3 *A geometric graph G is* ***well-distributed*** *if every convex area of size at least* $\frac{3\pi}{32}r^2$ *(in the unit square) has at least one node in it.*

In these graphs the nodes are distributed "nicely" across the unit square and in particular do not contains large "holes": empty convex regions with area larger than $\frac{3\pi}{32}r^2$. .

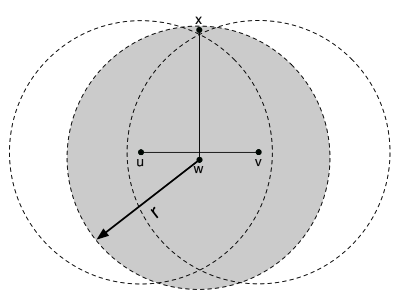

Fig. 2. A case where edges (w, x) and (u, v) are consistent and intersect in $LocalDel(G)$)

Lemma 5 *If* G*, is a well-distributed geometric graph, then consistent edges do not intersect in* $LocalDel(G)$*.*

Proof: Assume (u, v) and (w, x) are two consistent edges that intersect in $LocalDel(G)$, by Proposition 1 we can remove all nodes from the graph and consider only these two edges that must still exist and intersect. From Lemma 4.1 in [14] it must be the case that one of the four nodes is a neighbor of all other three, w.l.o.g let it be w. If any of the other three nodes sees all the four nodes it must be the case that either u or v sees all four of them, w.l.o.g let it be u. But in this case since w and u see all four nodes $T(w) = T(u)$ and either (w, x) and (u, v) do not intersect or at least one of them is inconsistent which leads to contradiction. So assume w is the only node that sees all the other four. Note that since w selected (w, x) as an edge while having information on the four nodes, (u, v) is a the non-Delaunay edge among the two. Observe that x must be outside $disk(u) \cup disk(v)$ (otherwise u or v see the four nodes), so it must be the case that $||wx|| \geq \frac{\sqrt{3}}{2}r$, since $||uv||$ is at most r and the edges intersect by assumption, see Fig. 2. Note also that since u and v choose (u, v) as an edge in $LocalDel(G)$ we must have that the circumcircle $disk(u, v, w) \cap (disk(u) \cup disk(v))$ is empty. In particular, this imply that the disk D of diameter $\frac{\sqrt{3}}{2}r$ which is tangent to the midpoint between u and v is empty. (see the gray disk in Fig 3). Since w, x, u, v

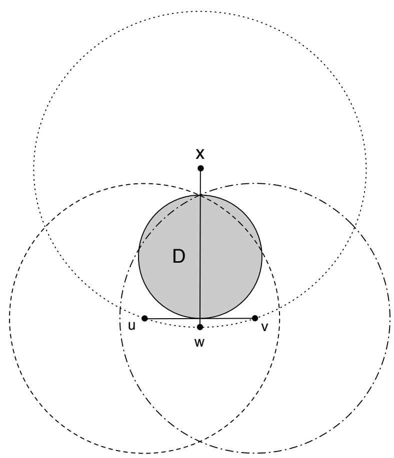

Fig. 3. A disk D that must be included in the area $disk(u, v, w) \cap (disk(u) \cup disk(v))$

are all in the unit square, it must be the case that at least half of D is inside the unit square as well. Now since G is *well-distributed* and the area of half of D is $\frac{3\pi}{32}r^2$ there is at least one node in that half, contradicting the consistency of (u, v) in $LocalDel(G)$ □

This lemma stands behind the core of our algorithm; For well distributed graph G all one needs to do to get $RDG(G)$ is to remove all inconsistent edges. Note, however, that it is still the case that even for well-distributed G, there may be inconsistent edges in $LocalDel(G)$. As Fig. 4 illustrates an edge (u, v) can be inconsistent at v since the area $disk(v) \cap disk(u, v, w)$ (the gray area in the figure) can become arbitrary small next to the boundaries of the unit square. (For a similar reason it can be shown that long Delaunay edges can exist in $Del(G)$, and in particular on the convex hull of V). Before formally proving the

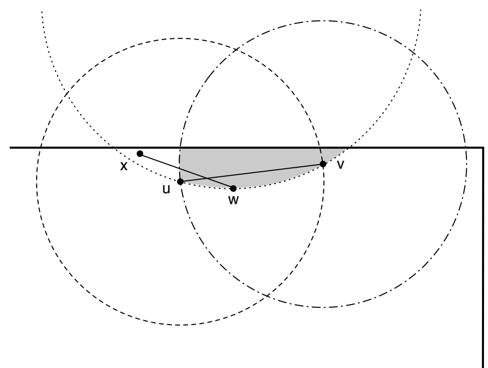

Fig. 4. An example where inconsistent edge (u, v) exist next to the border of the unit square

correctness of Algorithm 1 we need to show that random geometric graphs are well-distributed. We will do so by utilize a coupon collector argument.

Lemma 6 *If* $r \geq \sqrt{\frac{32}{3}} r_{\text{con}}$ *then w.h.p* $\mathcal{G}(n, r)$ *is well-distributed.*

Proof: Recall that $r_{\text{con}}^2 = \frac{\log n + \gamma_n}{\pi n}$ and $\gamma_n \to \infty$. Partition the unit square into convex *bins* of size $\frac{3\pi}{32} r^2$ so the number of bins B is

$$\frac{32}{3\pi r^2} = \frac{32}{3\pi} \frac{3\pi n}{32(\log n + \gamma_n)} = \frac{n}{\log n + \gamma_n}$$

Now it is a known result [20] that if one throws balls, uniformly at random into B bins, the expected number of balls needed to fill every bin with at least one ball is $B \log B$.If we want the result with high probability then one need to throw $B \log B + \gamma_n B$ balls. To conclude the proof we need to show that $n \geq B \log B + \gamma_n B$.

$$\begin{aligned} B \log B + \gamma_n B &= \frac{n}{\log n + \gamma_n} \log \left(\frac{n}{\log n + \gamma_n} \right) + \gamma_n \frac{n}{\log n + \gamma_n} \\ &= n \left(\frac{\log n}{\log n + \gamma_n} - \frac{\log(\log n + \gamma_n)}{\log n + \gamma_n} + \frac{\gamma_n}{\log n + \gamma_n} \right) \\ &= n(1 - \frac{\log(\log n + \gamma_n)}{\log n + \gamma_n}) \\ &\leq n \end{aligned}$$

□

Now we can proceed and prove Algorithm 1 correctness.

Proof of Theorem 1: From the last lemma. w.h.p the $\mathcal{G}(n,r)$ in the theorem is well-distributed. From Lemma 3 the algorithm removes all inconsistent edges after step 4. From Lemma 2 the resulting graph is a super graph of $UDel(G)$ and from Lemma 5 the resulting graph is a planar. □

5.2 Bounding the number of messages

Let $I = [\frac{\sqrt{3}}{4}r, 1 - \frac{\sqrt{3}}{4}r]^2$ be the inner square centered in the unit square where each side of I is at distance $\frac{\sqrt{3}}{4}r$ from the side of the unit square. For a well distributed G we have:

Lemma 7 *If $u \in I$ and $(u,v) \in LocalDel(G)$ then $||uv|| < \frac{\sqrt{3}}{4}r$.*

Proof: Let $(u,v) \in LocalDel(G)$ and $u \in I$. Assume $||uv|| \geq \frac{\sqrt{3}}{4}r$, then the area of each half of $disk(u,v)$ is at least $\frac{3\pi}{32}r^2$ and at least this area of each half is inside the unit square. Since G is well distributed each half contains at least one node, so (u,v) can't be an edge in $LocalDel(G)$. Contradiction. □

Lemma 8 *If $(u,v) \in LocalDel(G)$ and $u,v \in I$ then the edge (u,v) is consistent.*

Proof: Let $(u,v) \in LocalDel(G)$ and $u,v \in I$. Assume (u,v) is inconsistent and w.l.o.g assume it is local inconsistent at u. Then u must have a *proof* for the inconsistency, let it be $\triangle uwx \in T(u)$. Now w or x must be in I so from previous lemma $||wx|| < \frac{\sqrt{3}}{4}r$. Since $||uv||$ is also less than $\frac{\sqrt{3}}{4}r$ both x,w are in $N(u,v)$ and $\triangle uwx$ cannot be a *proof*. Contradiction. □

Now we can also prove the upper bound on the number of messages.

Proof of Theorem 2: As before $\mathcal{G}(n,r)$ is well-distributed w.h.p. There is only one step of communication and messages are sent only from nodes with local inconsistent edges. From Lemma 8 only edges $\{(u,v) \mid u,v \notin I\}$ can be inconsistent. The result follows since there are $\Theta(\sqrt{n \log n})$ nodes outside I and each sends at most 6 proofs. □

Using bins we can show that there are $\Theta(\sqrt{n/\log n})$ bins outside I, each with $\Theta(\log n)$ nodes. We conjecture that only constant number of inconsistent edges exist between neighboring bins (note that each bin is a clique so a bin cannot contain inconsistent edge) and state the following:

Conjecture 1 *For $r \geq \sqrt{\frac{32}{3}}r_{\text{con}}$ the number of messages in Algorithm 1 is $O(\sqrt{\frac{n}{\log n}})$*

Figure 5 shows early results of the number of messages in Algorithm 1 for different sizes of random networks, ranging from 100 to 3200 nodes. We choose r to be the same as in the theorems and plot the average number of messages for 10 different runs. These results are compared with the plot of $n/\log n$.

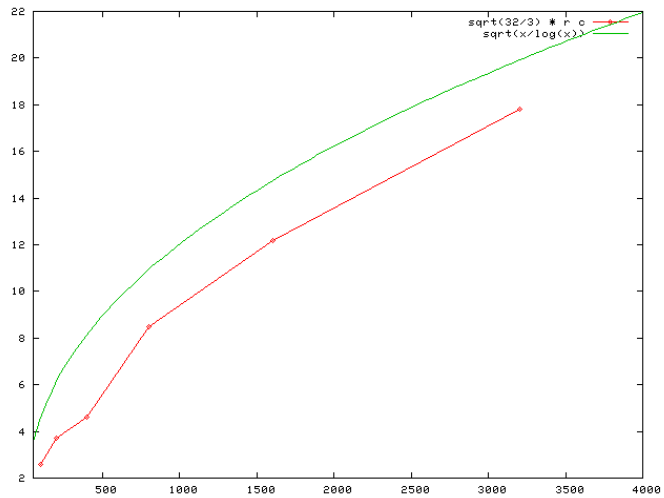

Fig. 5. Average number of messages in Algorithm 1 for different size random networks.

6 Conclusion

We offer a novel local algorithm to construct planar spanner in random wireless networks. Previous algorithms sent a message for each triangle in the Restricted Delaunay Graph, and in particular by the node with the largest angle, On the other hand, our algorithm avoids sending unnecessary messages far from the boundary and thus reduces the total number of messages from $\Theta(n)$ to $O(\sqrt{n \log n})$. Moreover, our results are stated in terms of *well-distributed* graphs, deterministic or random, and thus can be applied to more general graphs then the ones discussed here.

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