On The Cover Time of Random Geometric Graphs

Chen Avin and Gunes Ercal

Computer Science Department
University of California, Los Angeles
Los Angeles, CA 90095-1596, USA.
{avin, ercal}@cs.ucla.edu

Abstract. The cover time of graphs has much relevance to algorithmic applications and has been extensively investigated. Recently, with the advent of ad-hoc and sensor networks, an interesting class of random graphs, namely *random geometric graphs*, has gained new relevance and its properties have been the subject of much study. A random geometric graph $\mathcal{G}(n,r)$ is obtained by placing n points uniformly at random on the unit square and connecting two points iff their Euclidean distance is at most r. The phase transition behavior with respect to the radius r of such graphs has been of special interest. We show that there exists a critical radius r_{opt} such that for any $r \geq r_{\text{opt}}$ $\mathcal{G}(n,r)$ has optimal cover time of $\Theta(n \log n)$ with high probability, and, importantly, $r_{\text{opt}} = \Theta(r_{\text{con}})$ where r_{con} denotes the critical radius guaranteeing asymptotic connectivity. Moreover, since a disconnected graph has infinite cover time, there is a phase transition and the corresponding threshold width is $O(r_{\text{con}})$. We are able to draw our results by giving a tight bound on the electrical resistance of $\mathcal{G}(n,r)$ via the power of certain constructed flows.

1 Introduction

The *cover time* C_G of a graph G is the expected time taken by a simple random walk on G to visit all nodes in G. This property has much relevance to algorithmic applications [1–5], and methods of bounding the cover time of graphs have been thoroughly investigated [6–11]. Several bounds on the cover times of particular classes of graphs have been obtained with many positive results [8, 9, 12–14].

A random geometric graph (RGG) is a graph $\mathcal{G}(n,r)$ resulting from placing n points uniformly at random on the unit square[1] and connecting two points iff their Euclidean distance is at most r. While these graphs have traditionally been studied in relation to subjects such as statistical physics and hypothesis testing [15], random geometric graphs have gained new relevance with the advent of ad-hoc and sensor networks [16, 17] as they are a model of such networks. Sensor networks have strict energy and memory constraints and in many cases are subject to high dynamics, created by failures, mobility and other factors.

[1] We focus on the 2-dimensional, see section 6 for discussion

Thus, purely deterministic algorithms have disadvantages for such networks as they need to maintain data structures and have expensive recovery mechanism. Recently, questions regarding the random walk properties of such networks have been of interest especially due to the locality, simplicity, low-overhead and robustness to failures of the process [18–20]. In particular random walk techniques have been proposed for gossiping in random geometric graphs [1], for information collection and query answering [21, 5] and even for routing [22, 23].

In ad-hoc and sensor networks, interference grows with increased communication radius. So, for a desirable property P of random geometric graphs, one wants to find a tight upper bound on the smallest radius r_P, that will guarantee that P holds with high probability. The radius r_P is called *critical radius* if P exhibits a sharp threshold, the difference between the smallest radius for which the property holds with high probability and the radius for which the property holds with very low probability goes to zero as $n \to \infty$. The critical radius for connectivity, r_{con}, has been of special interest, and it has been shown that if $\pi r^2 \geq \pi r_{\text{con}}^2 = \frac{\log n + \gamma_n}{n}$ then $\mathcal{G}(n,r)$ is connected with probability going to one as $n \to +\infty$ iff $\gamma_n \to +\infty$ [24, 25].

In this paper we study the existence of a critical radius r_{opt} that will guarantee with high probability that $\mathcal{G}(n,r)$ with $r \geq r_{\text{opt}}$ has *optimal cover-time*. That is cover time of $\Theta(n \log n)$ [26], the same order as the complete graph. We show that such a threshold does exist, and, surprisingly, occurs at a radius $r_{\text{opt}} = \Theta(r_{\text{con}})$.

1.1 Discussion of Our Results and Techniques

Our main result can be formalized as follows:

Theorem 1 (Cover Time of RGG). *For $c > 1$, if $r^2 \geq \frac{c8 \log n}{n}$, then w.h.p.*[2] *$\mathcal{G}(n,r)$ has cover time $\Theta(n \log n)$. If $r^2 \leq \frac{\log n}{\pi n}$, then $\mathcal{G}(n,r)$ has infinite cover time with positive probability (bounded away from zero).*

The main contribution of this paper is in giving new tight theoretical bounds on the cover time and sharp threshold width associated with cover time for random geometric graphs. Our results improve upon bounds on the cover time obtained through bounding the mixing-time and spectral gap of random geometric graphs [27, 20, 19], as cover time can be bounded by the spectral gap [9]. In particular, the spectral gap method only guarantees optimal cover time of $\mathcal{G}(n,r)$ for $r = \Theta(1)$.

Aside from that, our results also have important implications for applications. Corollaries to our results are that both the *partial cover time* [5], which is the expected time taken by a random walk to visit a constant fraction of the nodes, and the *blanket time* [28], which is the expected time taken by a random walk to visit all nodes with frequencies according to the stationary distribution, are optimal for random geometric graphs. This demonstrates both the efficiency and

[2] Event $\mathcal{E}_n$ occurs with high probability if probability $P(\mathcal{E}_n)$ is such that $\lim_{n\to\infty} P(\mathcal{E}_n) = 1$.

quality of random walk approaches and certain token-management schemes for some ad-hoc and sensor networks [29, 1, 5].

In a recent related work Goel *et al.* [30] have proved that any monotonic property of random geometric graphs has a sharp threshold and have bounded the threshold width. While for general graphs optimality of cover time is not a monotonic property (see full version [31]), it follows from our result that optimality of cover time is monotonic for $\mathcal{G}(n,r)$ and has a threshold width of $O(r_{\mathrm{con}})$.

The method that we used to derive our result is by bounding the electrical resistance of $\mathcal{G}(n,r)$, which bounds the cover time by the following result of Chandra *et al.* [8]: for any graph with n nodes and m edges, where R is the the electrical resistance of the graph:

$$mR \leq \textit{cover time} \leq O(mR\log n) \tag{1}$$

In turn, we bound the resistance R of $\mathcal{G}(n,r)$ by bounding the power of a unit flow as permitted by Thomson's Principle which we formalize later. For any pair of points u and v, we construct a flow c in such a manner that the power of the flow satisfies $P(c) = O(\frac{n}{m}) = O(\frac{1}{\delta_{\mathrm{avg}}})$ where δ_{avg} denotes the average degree of a node in $\mathcal{G}(n,r)$. Since $R \leq P(c)$ the above flow together with (1) establish to be sufficient for $\mathcal{G}(n,r)$ to have optimal cover time.

To construct a flow from u to v, we partition the nodes into contour layers based on distance from u and expanding outward until the midpoint between u and v, then from the midpoint line onward contracting towards v in a mirror fashion. The idea of using contour layers that expand with distance from a point is similar to the layering ideas used by Chandra *et al.* [8] for meshes and originally by Doyle and Snell [32] for infinite grids. Layers in our case can be visualized as slices of an isosceles right triangle along the hypotenuse that connects u and v. The flow can thus be thought of as *moving through* consecutive layers, with the total flow on the edges connecting consecutive layers being 1. Just as the variance of a probability function is minimized for the uniform distribution, we minimize the power by allocating flow almost uniformly along the set of edges used between layer l and layer $l+1$.

The construction of the above flow is based on "nice" properties of random geometric graphs, such as the uniformity of nodes distribution and the regularity of node degree. We formalize this "niceness" using the notion of a *geo-dense* graph: a geometric graph (random or deterministic) with close to uniform node density across the unit square. In *geo-dense* graphs there are no large areas that fail to contain a sufficient number of nodes. To construct the flow we define *bins* as equal size areas that partition the unit square. These bins are used as the building blocks of our layered flow: nodes in neighboring bins are in the same clique, and only edges between neighboring bins contribute to the flow. Finally, We show that *random* geometric graphs are in fact *geo-dense* for radius on the order of $\Theta(r_{\mathrm{con}})$. Note however that *geo-dense* graphs are not necessary dense graphs in the graph theoretic meaning, i.e have $\Theta(n^2)$ edges. For example RGG are *geo-dense* even with $\Theta(n\log n)$ edges.

1.2 Related Work

There is a vast body of literature on cover times and on geometric graphs, and to attempt to summarize all of the relevant work would not do it justice. We have already mentioned some of the related results previously, however, here we would like to highlight the related literature that has been most influential to our result, namely that of Chandra *et al.* [8] and Doyle and Snell [32].

The work of Doyle and Snell [32] is a seminal work regarding the connection between random walks and electrical resistance. In particular, they proved that while the infinite 2-dimensional grid has infinite resistance, for any $d \geq 3$ the resistance of the d-dimensional grid is bounded from above, and these results were established to be sufficient in re-proving Pólya's beautiful result that a random walk on the infinite 2-dimensional grid is recurrent whereas a random walk on the infinite d-dimensional grid for any $d \geq 3$ is transient. In obtaining this result, essentially thors bounded the power of a unit current flow from the origin out to infinity and found that the power diverges for the 2-dimensional case and converges for every dimension greater than two. The authors used a layering argument, namely partitioning nodes into disjoint contour layers based on their distance from the origin, and the rate of growth of consecutive layers can be seen as the crucial factor yielding the difference between the properties of the different dimensions. Later, Chandra *et al.* [8] proved the tight relation between commute time and resistance, and used that relationship to extend Doyle and Snell's result by bounding the cover time of the *finite* d-dimensional mesh by computing the power and resistance via an expanding contour layers argument. Together with the tight lower bound of Zuckerman [10], they showed that the 2-dimensional torus has cover time of $\Theta(n \log^2 n)$, and for $d \geq 3$ the d-dimensional torus has an optimal cover time of $\Theta(n \log n)$.

While this paper deals with random geometric graphs there are striking similarities between $\mathcal{G}(n,r)$ and a more familiar family of random graphs, the *Bernoulli* graphs $\mathcal{B}(n,p)$ in which each edge is chosen independently with probability p [33]. For example, for critical probability $p_{\text{con}} = \pi r_{\text{con}}^2 = \frac{\log n + \gamma_n}{n}$, $\mathcal{B}(n,p)$ is connected with probability going to one as $n \to +\infty$ iff $\gamma_n \to +\infty$, and both classes of graphs have sharp thresholds for monotone properties [33]. Regarding cover time, Jonasson [12] and Cooper and Frieze [14] gave tight bounds on the cover time and an interesting aspect of our result is that we add another similarity and both classes of graphs have optimal cover time around the same threshold for connectivity. Yet, despite the similarities between $\mathcal{G}(n,r)$ and $\mathcal{B}(n,p)$, *Bernoulli* graphs are not appropriate models for connectivity in wireless networks since edges are introduced independent of the distance between nodes. In wireless networks the event of edges existing between i and j and between j and k is *not* independent of the event of an edge existing between k and i. There are other notable differences between $\mathcal{G}(n,r)$ and $\mathcal{B}(n,p)$ as well. For example, the proof techniques for the above results for $\mathcal{G}(n,r)$ are very different than the proof techniques for the respective results for $\mathcal{B}(n,p)$. Interestingly, whereas the proof of [14] for optimality of cover time in Bernoulli graphs of $\Theta(\log n)$ average degree depends on the property that Bernoulli graphs do *not* have small cliques (and, in particular that small cycles are sufficiently far apart), in the case of random geometric graphs the existence of many small cliques uniformly distributed over the unit square like bins is essential in our analysis.

Another recent result with a bin-based analysis technique for random geometric graphs is that of Muthukrishnan and Pandurangan [34]. However, as their technique uses large overlapping bins where the overlap is explicitly stated to be essential and there is no direct utilization of cliques.

2 Bounding The Cover Time via Resistance

For a graph $G = (V, E)$ with $|V| = n, |E| = m$, the electrical network $\mathcal{E}(G)$ is obtained by replacing each edge $e \in E$ with a 1 Ohm resistor, and this is the network we analyze when we speak of the resistance properties of G. For $u, v \in V$ let R_{uv} be the *effective resistance* between u and v: the voltage induced between u and v by passing a current flow of one ampere between them. Let R be the electrical resistance of G: the maximum effective resistance between any pair of nodes [32].

Let H_{uv} be the *hitting time*, the expected time for a random walk starting at u to arrive to v for the first time, and let C_{uv} be the *commute time*, the expected time for a random walk starting at u to first arrive at v and then return to u. Chandra *et al.* [8] proved the following equality for the commute time C_{uv} in terms of the effective resistance R_{uv}:

Theorem 2. *For any two vertices* u *and* v *in* G *the commute time* $C_{uv} = 2mR_{uv}$

Using this direct relation between resistance and random walks and Matthews' theorem [6] they introduced the bound of (1) on the cover time for G.

Let $H_{\max}$ be the maximum hitting time over all pairs of nodes in G. Since $H_{uv} \leq C_{uv}$ it follows that $H_{\max} \leq \max_{u,v \in V} C_{uv} = 2mR$. In [5] it has been shown that the partial cover time can be bounded by $H_{\max}$, so combining:

$$\textit{partial cover time} \leq O(mR) \tag{2}$$

Thus, by bounding the resistance R we may obtain tight bounds on the cover time C_G through (1) and on the partial cover time through (2).

A powerful method used to bound resistance is by bounding the power of a current flow in the network. The following definitions and propositions from the literature [8, 32, 35] help to formalize this method.

Definition 1 (Power of a flow). *Given an electrical network* (V, E, ρ), *with resistance* $\rho(e)$ *for each edge* e, *a flow* c *from a* source u *to a* sink v *is a function from* $V \times V$ *to* $\mathbb{R}$, *having the property that* $c(x, y) = 0$ *unless* $\{x, y\} \in E$, *and* c *is anti-symmetric, i.e.,* $c(x, y) = -c(y, x)$. *The net flow out of a node will be denoted* $c(x) = \sum_{y \in V} c(x, y)$ *and* $c(x) = 0$ *if* $x \neq u, v$. *The flow along an edge* e *is* $c(e) = |c(u, v)|$. *The power* $P(c)$ *in a flow is* $P(c) = \sum_{e \in E} \rho(e) c^2(e)$. *A flow is a current flow if it satisfies Kirchoff's voltage law, i.e., for any directed cycle* $x_0, x_1, \ldots, x_{k-1}, x_0$, $\sum_{i=0}^{k-1} c(x_i, x_{i+1 \bmod k}) \cdot \rho(x_i, x_{i+1 \bmod k}) = 0$.

Proposition 1. *[Thomson Principle [32, 35]] For any electrical network* (V, E, ρ) *and flow c with only one source u, one sink v, and* $c(u) = -c(v) = 1$ *(i.e a* ***unit*** *flow), we have* $R_{uv} \leq P(c)$*, with equality when the flow is a current flow.*

Finally,

Proposition 2. *[Rayleigh's Short/Cut Principle [32]] Resistance is never raised by lowering the resistance on an edge, e.g. by "shorting" two nodes together, and is never lowered by raising the resistance on an edge, e.g. by "cutting" it.*

3 The Cover Time and Resistance of Geometric Graphs

Before proving Theorem 1 about *random* geometric graphs we are going to prove a more general Theorem about geometric graphs. A **geometric graph** is a graph $G(n, r) = (V, E)$ with $n = |V|$ such that the nodes of V are embedded into the unit square with the property that $e = (u, v) \in E$ if and only if $d(u, v) \leq r$ (where $d(u, v)$ is the Euclidean distance between points u and v). We say that a geometric graph (either random or deterministic) is **geo-dense** if every square bin of area at least $A = r^2/8$ (in the unit square) has $\Theta(nA) = \Theta(nr^2)$ nodes.

Theorem 3. *A geometric graph* $G(n, r)$ *that is* geo-dense *and has* $r = \Theta(\frac{\log n}{n})$ *has optimal cover time of* $\Theta(n \log n)$*, optimal partial cover time of* $\Theta(n)$*, and optimal blanket time of* $\Theta(n \log n)$*.*

Let $G(n, r)$ be a geometric graph that is *geo-dense*. We will prove Theorem 3 using the bound on the cover time from Eq. (1) and by bounding the resistance between any two points u, v in $G(n, r)$. Let V be the set of nodes of $G(n, r)$ and $\delta(v)$ denote the degree (i.e number of neighbors) of $v \in V$

Claim 1 $\forall v \in V \;\; \delta(v) = \Theta(nr^2)$

Proof. First note that the *geo-dense* property guarantees that if we divide the unit square into square bins of size $\frac{r}{\sqrt{2}} \times \frac{r}{\sqrt{2}}$ each, then the number of nodes in every bin will be $\Theta(nr^2)$. Since, for every bin, the set of nodes in the bin forms a clique, and every node $v \in V$ is in some bin, we have that $\delta(v) = \Omega(nr^2), \forall v \in V$. Similarly, when we divide the area into bins of size $r \times r$ every node may be connected to the nodes of at most nine bins (that is its own bin and the bordering bins), and we have that $\delta(v) = \Theta(nr^2), \forall v \in V$. □

Thus, since we showed that $m = |E| = \Theta(n^2 r^2)$, if the resistance R of $G(n, r)$ is $O(\frac{n}{m}) = O(\frac{1}{nr^2})$ then we are done.

Theorem 4. *The resistance* R_{uv} *between* $u, v \in V$ *is* $\Theta(\frac{1}{nr^2} + \frac{\log(d(u,v)/r)}{n^2 r^4})$*.*

Proof. The proof of the upper bound will be by bounding the power of a unit flow c that we construct between u and v.

Let $T(u, v)$ be an isosceles right triangle such that the line (u, v) is the hypotenuse. It is clear that such a triangle which lies inside the unit square must

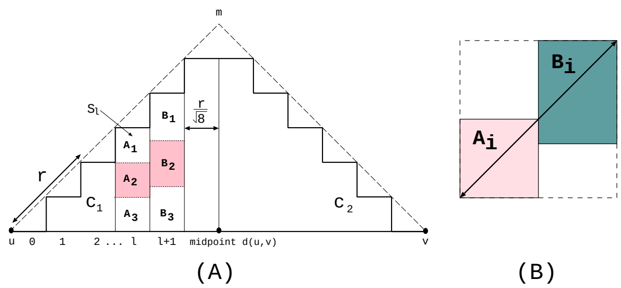

Fig. 1. $T(u, v)$ and the flow c between u and v in $G(n, r)$)

exist. We divide our flow c into two disjoint flows c_1 and c_2 where c_1 carries a unit flow from u up to the line perpendicular to the *midpoint* of $d(u, v)$ in increasing layer size, and c_2 forwards the flow in decreasing layer size up to v which is the only sink. By symmetry we can talk only about c_1 since the construction of c_2 mirrors that of c_1 and $P(c) = P(c_1+c_2) = 2P(c_1)$ since the flows are disjoint. To construct the flow in c_1 we divide the line $(u, midpoint(u, v))$ into $d(u, v)\sqrt{2}/r$ segments of size $r/\sqrt{8}$, and number them from 0 to $d(u, v)\sqrt{2}/r - 1$ (see Fig 1 (A))[3]. Let S_l be the largest rectangle of width $r/\sqrt{8}$ included in the intersection of the area perpendicular to the l^{th} segment and $T(u, v)$. S_l will define the l^{th} layer in our flow. Note that the area of S_l is $lr^2/8$ and contains l squares of area $r^2/8$, each of them containing $\Theta(nr^2)$ nodes by the *geo-dense* property.

Let $V_l \subseteq V$ be the set of nodes in layer l. $V_0 = u$, and for $l > 0$ a node v is in layer l if and only if it is located inside S_l. It follows that $|V_l| = \Theta(nr^2l)$. Edges in our flow are only among edges $e = (x, y)$ s.t. $x \in V_l$ and $y \in V_{l+1}$, and all other edges have zero flow. In particular, the set of edges E_l that carries flow from layer l to layer $l + 1$ in c_1 is defined as follows: for the case $l = 0$, E_0 contains all the edges from u to nodes in V_1, noting that $|E_0| = |V_1| = \Theta(nr^2)$ since $u \cup V_1$ is a clique (i.e the maximum $d(u, x), x \in V_1$ is r). This allows us to make the flow uniform such that each node in V_1 has incoming flow of $1/|V_1|$ and for each edge $e \in E_0$, $c_1(e) = 1/|E_0|$. For $l > 0$ (see again Fig. 1 (A)) we divide S_l into l equal squares $A_1, A_2, \ldots A_l$ each of size $r^2/8$. Let V_{A_i} be the set of nodes contained in the area A_i. We then divide S_{l+1} into l equal sized rectangles $B_1, B_2 \ldots B_l$ and define V_{B_i} similarly, with B_i touching A_i for each i.

Now let $E_l = \{(x, y)|x \in V_{A_i} \text{ and } y \in V_{B_i}\}$. Note again that since, for each i, the maximum $d(x, y)$ between nodes in A_i and nodes in B_i is r (see Fig. 1 (B)), $V_{A_i} \cup V_{B_i}$ is a clique (as the worst case distance occurs between the first two layers). So, the number of edges crossing from A_i to B_i is $|V_{A_i}||V_{B_i}| = \Theta(n^2r^4)$ by *geo-dense* property. The clique construction allows us to easily maintain the uniformity of the flow such that into each node in V_{B_i} the total flow is $1/l|V_{B_i}|$,

[3] Assume for simplicity the expression divides nicely, if not, the proof holds by adding one more segment that will end at the midpoint and overlap with the previous segment.

and each edge carries a flow of $\Theta(1/n^2r^4l) = \Theta(1/E_l)$. All other edges have no flow. Now we compute the power of c:

$$
\begin{aligned}
R_{uv} &\le \sum_{e\in c} c(e)^2 = \sum_{e\in c_1} c_1(e)^2 + \sum_{e\in c_2} c_2(e)^2 = \\
&= 2\sum_{l=0}^{\sqrt{2}d(u,v)/r} \sum_{e\in E_l} c_1(e)^2 = 2\frac{1}{|E_0|} + 2\sum_{l=1}^{\sqrt{2}d(u,v)/r} \frac{1}{|E_l|} \\
&= 2O(\frac{1}{nr^2}) + 2O(\frac{1}{n^2r^4}) \sum_{l=1}^{\sqrt{2}d(u,v)/r} \frac{1}{l} \\
&= O(\frac{1}{nr^2} + \frac{\log(d(u,v)/r)}{n^2r^4})
\end{aligned}
$$

To prove the lower bound we again follow in the spirit of [32] and use the "Short/Cut" Principle. We partition the graph into $\lfloor d(u,v)/r \rfloor + 1$ partitions by drawing $\lfloor d(u,v)/r \rfloor$ squares perpendicular to the line (u,v), where the first partition P_0 is only u itself and the l^{th} partition P_l is the area of the l^{th} square excluding the $(l-1)^{\text{th}}$ square area. The last partition contains all the nodes outside the last square including v (see Fig 2 (A)). We are shorting all vertices in the same partition (see Fig. 2 (B), and following the reasoning of the upper bound, let m_l be the number of edges between partition l and $l+1$. m_0 is $\Theta(nr^2)$ and for $l > 0$, $m_l = \Theta(n^2r^4l)$, so

$$
\begin{aligned}
R_{uv} &\ge \sum_{l=0}^{\lfloor d(u,v)/r \rfloor} \frac{1}{m_l} \\
&= \Omega(\frac{1}{nr^2}) + \sum_{l=1}^{\lfloor d(u,v)/r \rfloor} \Omega(\frac{1}{n^2r^4l}) \\
&= \Omega(\frac{1}{nr^2} + \frac{\log(d(u,v)/r)}{n^2r^4})
\end{aligned}
$$

□

Corollary 1. *The resistance R of $G(n,r)$ is $\Theta(\frac{1}{nr^2} + \frac{\log(\sqrt{2}/r)}{n^2r^4})$.*

This follows directly from the fact that $\max d(u,v) \le \sqrt{2}$. Now we can prove Theorem 3.

Proof (of Theorem 3). Remember that $m = \Theta(n^2r^2)$, so all we need is $R = O(n/m) = O(1/nr^2)$ and then the cover time bound will follow by (1), the partial cover time bound will follow from (2), and the blanket time will follow from [28] and the $\log n$ order difference between the cover time and maximum hitting time. In order to have $R = \Theta(\frac{1}{nr^2})$ we want that $\frac{\log(\sqrt{2}/r)}{n^2r^4} = O(\frac{1}{nr^2})$, which means $\frac{\log(1/r)}{nr^2} \le \alpha$ for some constant α. Taking $r^2 = \frac{c\log n}{n}$, for a constant β, we get $\frac{\log(n/\beta\log n)}{\beta 2\log n} = \frac{1}{2\beta} - \frac{\log(\beta\log n)}{2\beta\log n} \le \frac{1}{2\beta}$. □

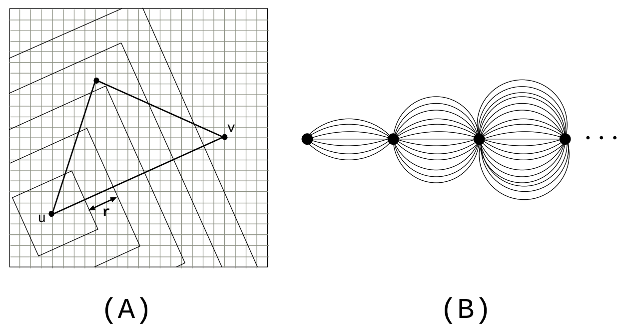

Fig. 2. Lower bound for R_{uv} on the $G(n, r)$

4 Cover Time and Resistance of $\mathcal{G}(n, r)$

After Proving Theorem 3, in order to prove Theorem 1 all we need to show is that for $c > 1$, $r^2 = \frac{c8logn}{n}$ is sufficient to guarantee with high probability that $\mathcal{G}(n, r)$ is *geo-dense*. Note however that the second part of the theorem follows directly from [25] since if $\mathcal{G}(n, r)$ is disconnected with positive probability bounded away from zero when $r^2 \leq \frac{\log n}{\pi n}$, then it has infinite cover time with at least the same probability.

To prove the *geo-dense* property for $\mathcal{G}(n, r)$ we utilize the following lemma which seems to be folklore [28] although we include a proof in the full version since we have not found a reference including a proof of the minimum condition.

Lemma 2 (Balls in Bins). *For a constant $c > 1$, if one throws $n \geq cB \log B$ balls uniformly at random into B bins, then w.h.p both the minimum and the maximum number of balls in any bin is $\Theta(\frac{n}{B})$.*

And, the following lemma easily follows from the Balls in Bins Lemma:

Lemma 3 (Node Density). *For constants $c > 1$ and $a \geq 1$, if $r^2 = \frac{ca \log n}{n}$ then w.h.p any area of size r^2/a in $\mathcal{G}(n, r)$ has $\Theta(c \log n)$ nodes.*

Proof. Let an area of r^2/a be a bin. If we divide the unit square into such equal size bins we have $B = \frac{n}{c \log n}$ bins. For the result to follow we check that Lemma 2 holds by showing that $n \geq c'B \log B$ for some constant $c' > 1$:

$$\begin{aligned} B \log B &= \frac{n}{c \log n} \log(\frac{n}{c \log n}) \\ &= \frac{n}{c \log n}(\log(n) - \log(c \log n)) \\ &= \frac{n}{c} - (\frac{n}{c \log n})(\log(c \log n)) \\ &\leq n/c \end{aligned}$$

□

Now combining the results of Lemmas (2) and (3) we can prove Theorem 1

Proof (of Theorem 1). Clearly from Lemma 3 for $c > 1$, $r^2 = \frac{c8\log n}{n}$ satisfies the *geo-dense* property w.h.p, and since r^2 is also $\Theta(\frac{\log n}{n})$ the result follows from Theorem 3. □

Corollary 2. *For $c > 1$, if $r^2 \geq \frac{c8\log n}{n}$, then w.h.p $\mathcal{G}(n, r)$ has optimal partial cover time $\Theta(n)$ and optimal blanket time $\Theta(n \log n)$.*

5 Deterministic Geometric Graphs

As an example of other applications of our results consider the following: for an integer k, let the k**-fuzz** [32] of a graph G be the graph G_k obtained from G by adding an edge xy if x is at most k hops away from y in G. In particular, let $G_1(n)$ denote the 2-dimensional grid of n nodes, and let $G_k(n)$ be the k-fuzz of $G_1(n)$. It is known that the cover time of $G_1(n)$ is $\Theta(n \log^2 n)$. and so we ask what is the minimum k s.t. G_k has an optimal cover time of $\Theta(n \log n)$. Using the results of Theorem 4 and the lower bound method of Zuckerman [10] we can prove the following (see [31]):

Theorem 5. *For any constant k, the cover time of $G_k(n)$ is $\Theta(k^{-2} n \log^2 n)$.*

Corollary 3. *$G_k(n)$ has Cover Time of $\Theta(n \log n)$ if $k = \gamma_n$ and $\lim_{n\to\infty} \dfrac{\log n}{\gamma_n^2} \leq c$ for some constant c.*

This means that if each node is neighbor with his $\Omega(\log n)$ closest neighbors, the cover time of the 2-dimensional grid becomes optimal.

6 Conclusions

We have shown that for a two dimensional random geometric graph $\mathcal{G}(n, r)$, if the radius r_{opt} is chosen just on the order of guaranteeing asymptotic connectivity then $\mathcal{G}(n, r)$ has optimal cover time of $\Theta(n \log n)$ for any $r \geq r_{\text{opt}}$. We present a similar proof for 1-dimensional random geometric graphs in [31]. We find that the critical radius guaranteeing optimal cover time is $r_{\text{opt}} = \Omega(\frac{1}{\sqrt{n}})$ for such graphs, whereas the critical radius guaranteeing asymptotic connectivity is $r_{\text{con}} = \frac{\log n}{n}$. So, unlike the 2-dimensional case, we have $r_{\text{opt}} = \omega(r_{\text{con}})$.

Our proof techniques can be generalized to the d-dimensional random geometric graph $\mathcal{G}^d(n, r)$, yielding that for any given dimension d, $r_{\text{opt}} = \Theta(r_{\text{con}})$ with correspondingly optimal cover time. However, both grow exponentially with d which seems to be a consequence of a separation between average degree and minimum degree for higher dimensions rather than just an artifact of our method. Nevertheless, the case of dimension $d = 2$ is considered to be the hardest one [36]. This can intuitively be seen from the mesh results. The case for $d = 1$ (i.e the cycle) is easy to analyze. For $d > 2$ the cover time of the d-dimensional mesh is optimal [8], and we can show that for any k the cover time of the k-fuzz is also optimal. On the other hand, as we showed earlier, the cover time of the $k-$fuzz in 2 dimensions (i.e. $G_k(n)$) for constant k is not optimal making this the most interesting case.

Acknowledgment: The authors would like to thank Shailesh Vaya and Eli Gafni for helpful discussions and David Dayan-Rosenman and the anonymous reviewers for their comments and corrections. The first author acknowledges partial support from ONR (MURI) grant #N00014-00-1-0617 and from the Department of Communication System Engineering at Ben-Gurion University, Israel.

References

1. Kempe, D., Dobra, A., Gehrke, J.: Gossip-based computation of aggregate information. In: Proc. of the 44th Annual IEEE Symposium on Foundations of Computer Science. (2003) 482–491
2. Gkantsidis, C., Mihail, M., Saberi, A.: Random walks in peer-to-peer networks. In: in Proc. 23 Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM). (2004)
3. Wagner, I.A., Lindenbaum, M., Bruckstein, A.M.: Robotic exploration, brownian motion and electrical resistance. Lecture Notes in Computer Science **1518** (1998) 116–130
4. Jerrum, M., Sinclair, A.: The markov chain monte carlo method: an approach to approximate counting and integration. In: Approximations for NP-hard Problems, Dorit Hochbaum ed. PWS Publishing, Boston, MA (1997) 482–520
5. Avin, C., Brito, C.: Efficient and robust query processing in dynamic environments using random walk techniques. In: Proc. of the third international symposium on Information processing in sensor networks. (2004) 277–286
6. Matthews, P.: Covering problems for Brownian motion on spheres. Ann. Probab. **16** (1988) 189–199
7. Aldous, D.J.: Lower bounds for covering times for reversible Markov chains and random walks on graphs. J. Theoret. Probab. **2** (1989) 91–100
8. Chandra, A.K., Raghavan, P., Ruzzo, W.L., Smolensky, R.: The electrical resistance of a graph captures its commute and cover times. In: Proc. of the 21st annual ACM symposium on Theory of computing. (1989) 574–586
9. Broder, A., Karlin, A.: Bounds on the cover time. J. Theoret. Probab. **2** (1989) 101–120
10. Zuckerman, D.: A technique for lower bounding the cover time. In: Proc. of the twenty-second annual ACM symposium on Theory of computing. (1990) 254–259
11. Aleliunas, R., Karp, R.M., Lipton, R.J., Lovász, L., Rackoff, C.: Random walks, universal traversal sequences, and the complexity of maze problems. In: 20th Annual Symposium on Foundations of Computer Science (San Juan, Puerto Rico, 1979). IEEE, New York (1979) 218–223
12. Jonasson, J.: On the cover time for random walks on random graphs. Comb. Probab. Comput. **7** (1998) 265–279
13. Jonasson, J., Schramm, O.: On the cover time of planar graphs. Electronic Communications in Probability **5** (2000) 85–90
14. Cooper, C., Frieze, A.: The cover time of sparse random graphs. In: Proc. of the fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA-03). (2003) 140–147
15. Penrose, M.D.: Random Geometric Graphs. Volume 5 of Oxford Studies in Probability. Oxford University Press (2003)
16. Estrin, D., Govindan, R., Heidemann, J., Kumar, S.: Next century challenges: Scalable coordination in sensor networks. In: Proc. of the ACM/IEEE International Conference on Mobile Computing and Networking. (1999) 263–270

17. Pottie, G.J., Kaiser, W.J.: Wireless integrated network sensors. Communications of the ACM **43** (2000) 51–58
18. Goel, A., Rai, S., Krishnamachari, B.: Monotone properties have sharp thresholds in random geometric graphs. STOC slides (2004) http://www.stanford.edu/ sanat/slides/thresholdsstoc.pdf.
19. Avin, C., Ercal, G.: Bounds on the mixing time and partial cover of ad-hoc and sensor networks. In: Proceedings of the 2nd European Workshop on Wireless Sensor Networks (EWSN 2005). (2005) 1–12
20. Boyd, S., Ghosh, A., Prabhakar, B., Shah, D.: Gossip and mixing times of random walks on random graphs. Unpublished. http://www.stanford.edu/ boyd/reports/gossip_gnr.pdf (2004)
21. Sadagopan, N., Krishnamachari, B., Helmy, A.: Active query forwarding in sensor networks (acquire). To appear Elsevier journal on Ad Hoc Networks (2003)
22. Braginsky, D., Estrin, D.: Rumor routing algorthim for sensor networks. In: Proc. of the 1st ACM Int. workshop on Wireless sensor networks and applications. (2002) 22–31
23. Servetto, S.D., Barrenechea, G.: Constrained random walks on random graphs: routing algorithms for large scale wireless sensor networks. In: Proc. of the 1st Int. workshop on Wireless sensor networks and applications. (2002) 12–21
24. Penrose, M.D.: The longest edge of the random minimal spanning tree. The Annals of Applied Probability **7** (1997) 340–361
25. Gupta, P., Kumar, P.R.: Critical power for asymptotic connectivity in wireless networks. In Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming (1998) 547–566
26. Feige, U.: A tight lower bound on the cover time for random walks on graphs. Random Structures and Algorithms **6** (1995) 433–438
27. Rai, S.: The spectrum of a random geometric graph is concentrated. http://arxiv.org/PS_cache/math/pdf/0408/0408103.pdf (2004)
28. Winkler, P., Zuckerman, D.: Multiple cover time. Random Structures and Algorithms **9** (1996) 403–411
29. Dolev, S., Schiller, E., Welch, J.: Random walk for self-stabilizing group communication in ad-hoc networks. In: Proc. of the 21st IEEE Symposium on Reliable Distributed Systems (SRDS'02), IEEE Computer Society (2002) 70
30. Goel, A., Rai, S., Krishnamachari, B.: Sharp thresholds for monotone properties in random geometric graphs. In: Proc. of the thirty-sixth annual ACM symposium on Theory of computing. (2004) 580–586
31. Avin, C., Ercal, G.: On the cover time of random geometric graphs. Technical Report 040050, UCLA (2004) ftp://ftp.cs.ucla.edu/tech-report/2004-reports/040050.pdf.
32. Doyle, P.G., Snell, J.L.: Random Walks and Electric Networks. Volume 22. The Mathematical Association of America (1984)
33. Bollobás, B.: Random Graphs. Academic Press, Orlando, FL (1985)
34. Muthukrishnan, S., Pandurangan, G.: The bin-covering technique for thresholding random geometric graph properties. In: Proc. of the ACM-SIAM Symposium on Discrete Algorithms, to appear. (2005)
35. Synge, J.L.: The fundamental theorem of electrical networks. Quarterly of Applied Math. (1951) 113–127
36. Aldous, D., Fill, J.: Reversible markov chains and random walks on graphs. Unpublished. http://stat-www.berkeley.edu/users/aldous/RWG/ book.html (1997)