On the meridional structure of extra-tropical Rossby waves

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ABSTRACT

The common derivation of Rossby waves is based on the quasi-geostrophic approximation, which is usually assumed to be valid for planetary meridional scales of order 1000 kilometers. A simple non-harmonic approximation for extratropical Rossby waves on the sphere is proposed in which the meridional coordinate is a parameter instead of as a continuous variable. It is shown that, in contrast to the quasi-geostrophic solution, the meridional structure of these non-harmonic Rossby waves becomes irrelevant to first order for determining the dispersion relation in this theory. The proposed approximation reproduces accurately numerical results obtained by general circulation model, the MITgcm, and captures the latitudinal dependence of the phase speed of these waves, when starting from arbitrary initial meridional structure. The proposed theory yields explicit expressions for the dispersion relation and wave structure.
1. Introduction

Rossby waves (Rossby 1939), also known as “second class” waves (Margules 1893; Hough 1898), are usually derive based on quasi-geostrophic approximation on the $\beta$-plane (Rossby 1939; Gill 1982; Pedlosky 1987). Yet, there are exact solutions for Rossby waves both on the sphere (e.g., Longuet-Higgins 1968a) and on the $\beta$-plane (e.g., Gill 1982). The exact solutions are much more complicated than the simple quasi-geostrophic solutions, and thus received lesser attention—the quasi-geostrophic-approximation-based solutions are usually assumed to be valid for planetary scales of the order of a thousand kilometers.

According to the classical quasi-geostrophy approximation (Rossby 1939; Pedlosky 1987; Gill 1982), Rossby waves are expressed as harmonic (sinusoidal) waves in the zonal and meridional directions where their frequency (and hence the phase speed) is determined by the values of the Coriolis parameters $f_0$ and $\beta$ in the middle of the domain. It is thus clear that according to the quasi-geostrophic Rossby wave solution, an initial eigenmode should propagates zonally and uniformly along a latitude circle as long as the conditions for the validity of the quasi-geostrophic approximation are satisfied. However, it is known in the community of ocean general simulation modelers, as well as from observations (see figure 4 of Chelton and Schlax (1996) and also Chelton et al. (2007)) that different meridional sections of a single perturbation propagate with different zonal phase speeds; see further discussion and demonstrations in Appendix A. This phenomenon is sometimes referred to as “$\beta$-dispersion” (see, Schopf et al. 1981).

The main goal of this study is to present a simple approximation for extra tropical Rossby waves that accounts for the latitudinal variations of Rossby waves frequency that is valid for
latitudinal scales of tens of degrees. The proposed approximation provides the time evolution for an arbitrary initial condition in the meridional direction. For this purpose, we use the linearized shallow water equations (LSWE) on the sphere; in the new approximation, to a zero order, the meridional coordinate acts like a parameter, allowing a simple derivation of the wave structure and dispersion relation. The suggested approximations are successfully validated by running the Massachusetts Institute of Technology general circulation model (MITgcm, Adcroft et al. 2002).

The paper is organized as follows: In Section 2 we briefly describe the numerical model (MITgcm) that is used in this study. A new approximation for the extra-tropical Rossby waves is suggested in Section 3. The numerical results are described in Section 4. A summary and discussion is then followed (Section 5).

2. Numerical model

We use the MITgcm (Adcroft et al. 2002; Marshall et al. 1997a, 1997b) for the numerical simulation. The MITgcm solves the primitive equations for the ocean (and possibly the atmosphere), and is a free surface, finite volume, $z$-coordinate model with shaved/partial bottom cells. Various state-of-the-art subgrid parametrizations and boundary layer schemes may be used in this model. The model was used in the past to study systems over a wide range of temporal and spatial (from scales of several degrees down to a few meters) scales (Adcroft et al. 2002).

Here we study the dynamics of Rossby waves in a channel with flat bottom on the sphere. The numerical configuration was chosen accordingly: one vertical layer of depth $H = 500$
m over 360° in that zonal direction and from 80°S to 80°N. The gravitation constant in all simulations is $g=0.01 \text{ ms}^{-2}$, a value that together with the channel depth $H$ results in a gravity wave speed of $\sqrt{gH} \approx 2.2 \text{ m/s}$, a typical value for first baroclinic mode phase speed. Various simulations over several narrower channel widths (with various meridional resolution) on the $\beta$-plane were also studied (see Appendix A). The resolution in the zonal and meridional directions is $1° \times 1°$. The initial conditions of the different examples are described below. Periodic boundary conditions are assumed in the zonal direction and free slip on the channel walls and at the bottom. The water density is set to be constant. The implicit free surface scheme of MITgcm was used in the simulations. The numerical simulations include the advection and viscosity terms. The integration time step is one hour. The horizontal viscosity coefficient is 300 m$^2$/s although we used other coefficient values to check the sensitivity of the results to the changes in this parameter. Other details related to the numerical simulations are described in Sec. 4 and Appendix A.

3. A new approximation for Rossby waves on the sphere

There is an infinite number of vertical modes of density stratification in the ocean (Gill 1982). The barotropic mode is very fast and is therefore more difficult to study while the first baroclinic mode is sufficiently slow (although it is the fastest of all baroclinic modes) to be the dominant mode of the baroclinic modes (Chelton et al. 1998).

The shallow water equations may be used to describe the dynamics of the first baroclinic
mode waves. For a flat channel of depth $H$ on the sphere the LSWE are (Gill 1982):

\[
\begin{align*}
\frac{\partial u}{\partial t} - 2\Omega \sin \phi v &= -\frac{g}{a \cos \phi} \frac{\partial \eta}{\partial \lambda} \\
\frac{\partial v}{\partial t} + 2\Omega \sin \phi u &= -\frac{g}{a} \frac{\partial \eta}{\partial \phi} \\
\frac{\partial \eta}{\partial t} + \frac{H}{a \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{H}{a} \frac{\partial v}{\partial \phi} - \frac{H}{a} v \tan \phi &= 0,
\end{align*}
\]

where $u$ and $v$ are the zonal and meridional velocities, $\eta$ is the free surface height, $g$ is the (reduced) gravity constant, $a$ is the Earth’s radius, and $\Omega$ is the Earth’s angular frequency. Here we neglect the nonlinear advection terms since they are very small due to the small velocities (small Rossby number) and ditto the viscosity terms. We assume periodic boundary conditions in the zonal direction and zero meridional velocity in at the channel walls.

Eqs. (1) may be reduced to a single equation in $\eta$ following a procedure similar to that given in Pedlosky (2003). First, it is necessary to express the velocities, $u$, $v$ in terms of $\eta$ as follows:

\[
\begin{align*}
\Re u &= -\frac{g}{a} \left( \frac{1}{\cos \phi} \frac{\partial^2 t}{\partial \lambda} + f \frac{\partial^2}{\partial \phi} \right) \eta \\
\Re v &= \frac{g}{a} \left( \frac{f}{\cos \phi} \frac{\partial^2 \eta}{\partial \lambda} - \frac{\partial \phi}{\partial \phi} \right) \eta,
\end{align*}
\]

where the operator $\Re$ is

\[
\Re = \frac{\partial^2 t}{\partial t^2} + f^2,
\]

where $f = 2\Omega \sin \phi$ is the Coriolis parameter. After a few algebraic operations a partial differential equation for $\eta$ is obtained:

\[
\left\{ \cos \phi \frac{\partial \Re}{\partial \lambda} - \frac{g H}{a^2} \left[ 2\Omega \cos \phi f^2 \frac{\partial \lambda}{\partial \lambda} + \Re \frac{1}{\cos \phi} \frac{\partial \lambda}{\partial \lambda} - 4\Omega \cos^2 \phi f \frac{\partial \phi}{\partial \phi} + \cos \phi \frac{\partial \phi}{\partial \lambda} + \Re \frac{\partial \phi}{\partial \phi} (\cos \phi \frac{\partial \phi}{\partial \phi}) \right] \right\} \eta = 0.
\]
It is possible to write Eq. (4) using only one dimensionless parameter \( \varepsilon \)

\[
\varepsilon = \frac{gH}{(2\Omega a)^2}
\]

and by scaling \( t \) and \( \lambda \) using this parameter \( \lambda = \sqrt{\varepsilon} \lambda \) and \( t = \hat{t}/(2\Omega \sqrt{\varepsilon}) \) where the “hat” symbol denotes dimensionless variables. The obtained equation is:

\[
\left\{ \cos \phi \left( \partial_{\hat{t}} \hat{R}^2 - \sin^2 \phi \partial_{\lambda \hat{t}} \right) - \hat{R} \frac{1}{\cos \phi} \partial_{\lambda \hat{t}} \varepsilon \left[ 2 \cos^2 \phi \sin \phi \partial_{\hat{t} \phi} + \cos \phi \partial_{H \lambda} \hat{R} \partial_{\phi} \left( \cos \phi \partial_{\hat{t} \phi} \right) \right] \right\} \eta = 0,
\]

where

\[
\hat{R} = \varepsilon \partial_{\hat{t} \hat{t}} + \sin^2 \phi.
\]

The boundary of zero meridional velocity on the channel walls may be expressed using Eq. (2)

\[
-\frac{1}{2\Omega} \partial_{t \phi} \eta + \tan \phi \partial_{\lambda \eta} \eta = 0.
\]

In the non-dimensional boundary conditions the factor \( 1/(2\Omega) \) is replaced by \( \varepsilon \). Note that only \( \lambda \) and \( t \) are rescaled while \( \phi \) remains as it is; \( \eta \) may be rescaled using an arbitrary constant. Also note that \( \varepsilon \) is the inverse of the nondimensional parameter used in some previous studies (Margules 1893; Hough 1898; Longuet-Higgins 1968a). In addition, a similar scaling applied for the \( \beta \)-plane LSWE yields a nondimensional parameter \( \varepsilon \) that is the square of the corresponding small parameter of the quasi-geostrophic approximation; the scaling for the LSWE on the \( \beta \)-plane is \( \hat{t} = t/\beta R, \hat{x} = R x, \hat{y} = f_0 y/\beta \), where \( R = \sqrt{gh}/f_0 \) and the small parameter is \( \varepsilon = (\beta R/f_0)^2 \). For typical values of \( g = 0.01 \text{ m/s}^2 \) and \( H = 500 \text{ m} \) we obtain \( \varepsilon \approx 5.8 \times 10^{-6} \ll 1 \). In the scaling presented above, the dimensional time variable is multiplied by \( 2\Omega \sqrt{\varepsilon} \) to yield the non-dimensional time variable, i.e., time is squeezed; this is
since Rossby waves are slow compared to the Earth rotation. In addition, the original zonal coordinate $\lambda$ is divided by $\sqrt{\varepsilon}$, meaning “zonal stretching”. This scaling is motivated by observations (and the numerical results presented below) that indicate that Rossby waves propagate almost entirely in the zonal direction, indicating much faster wave propagation in the zonal direction; this zonal stretching aims in “slowing” this propagation. In practice the above scaling is equivalent to the assumption of long (or large scale) meridional waves (or perturbations).

The free surface $\eta$ may be approximated expanded in power of $\varepsilon$

$$\eta = \eta_0 + \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \ldots \quad (9)$$

Eq. (6) then reduces to a simple equation for the zero order approximation for the free surface, $\eta_0$:

$$\left[ \frac{1}{\sin^2 \phi} \frac{\partial}{\partial \lambda} \left( \frac{1}{\cos^2 \phi} \frac{\partial}{\partial \lambda} + 1 \right) \right] \eta_0 = 0. \quad (10)$$

Eq. (10) does not contain derivatives with respect to the meridional variable $\phi$ and, thus, $\phi$ becomes a mere parameter. This fact enables the simple solution (shown below) that includes the meridional dependence of the wave frequency.

Assuming a wave solution of the form $\eta_0 = \exp \left[ i (k_\lambda \lambda - \dot{\omega} t + \delta) \right] \bar{\eta}_0(\phi) = \exp (i \theta) \bar{\eta}_0(\phi)$, one obtains the following dispersion relation

$$\dot{\omega}(\phi) = -\frac{k_\lambda \cos^2 \phi}{\sin^2 \phi \cos^2 \phi + k_\lambda^2}, \quad (11)$$

where the function $\bar{\eta}_0(\phi)$ may be any smooth function that satisfies the boundary conditions.

The dimensional dispersion relation is

$$\omega(\phi) = -\frac{2 \Omega \varepsilon k_\lambda \cos^2 \phi}{\sin^2 \phi \cos^2 \phi + \varepsilon k_\lambda^2}. \quad (12)$$
The derived dispersion relation and free surface were obtained for the extra-tropics as in the operator $\mathcal{R}$ the Coriolis parameter is assumed to be much larger than the time derivatives (and hence the Rossby wave frequency), as for the extra-tropics. Still, the dispersion relation seems to be valid other both tropical and extra-tropical regions under the assumption of long meridional waves.

At this stage, the zero order solution is known as the free surface was the only variable in Eq. (6), which is the equivalent equation to the shallow water equations. Once $\eta_0$ is found, it is possible to use it in Eq. (2) to obtain two uncoupled ordinary differential equations for the velocities, $u$ and $v$, where each of these equations is in essence a forced harmonic oscillator equation. Alternatively, it is possible to apply the above scaling in Eq. (2) to obtain

\[
\begin{align*}
\hat{\mathcal{R}} \hat{u} &= - \left( \frac{1}{\cos \phi} \partial_{\hat{t} \hat{\lambda}} + \sin \phi \partial_{\hat{\phi}} \right) \hat{\eta} \\
\hat{\mathcal{R}} \hat{v} &= \left( \tan \phi \partial_{\hat{\lambda}} - \varepsilon \partial_{\hat{t} \hat{\phi}} \right) \hat{\eta},
\end{align*}
\]

(13)

where the implied scaling for $\eta$, $u$, and $v$ is

\[
\eta = H \hat{\eta}, \quad u = 2 \Omega a \varepsilon \hat{u}, \quad v = 2 \Omega a \sqrt{\varepsilon} \hat{v}.
\]

(14)

Thus, to the zeroth order in $\varepsilon$ the non-dimensional velocities become

\[
\begin{align*}
\hat{u}_0 &= \frac{1}{\sin \phi} \left( \hat{t} \cos \theta \hat{\eta}_0 \hat{\omega}' - \sin \theta \hat{\eta}'_0 - \frac{\hat{\omega} \hat{k}_\lambda \sin \theta \hat{\eta}_0}{\sin \phi \cos \phi} \right), \\
\hat{v}_0 &= \frac{1}{\sin \phi} \frac{\hat{k}_\lambda \cos \theta \hat{\eta}_0}{\cos \phi}.
\end{align*}
\]

(15)

(16)

The zero order dimensional velocities are then

\[
\begin{align*}
u_0 &= \frac{g}{a f} \left( \hat{t} \cos \theta \hat{\eta}_0 \hat{\omega}' - \sin \theta \hat{\eta}'_0 - \frac{\omega k \lambda \sin \theta \hat{\eta}_0}{f \cos \phi} \right), \\
v_0 &= \frac{g}{a f} \frac{k \lambda \cos \theta \hat{\eta}_0}{\cos \phi}.
\end{align*}
\]

(17)

(18)
For long planetary waves \((k\lambda \sim 1)\), the last term in Eq. (17) may be neglected. This case may be associated with the planetary geostrophy solution. It follows from Eq. (16) that the boundary conditions \(v_0 = 0\) at channel walls are also satisfied by the free surface \(\eta_0\), i.e., \(\eta_0 = 0\) at the channel walls. Note that \(u_0\) does not have to vanish at the channel wall even if \(\eta_0\) does, since it includes, according to Eq. (17), the derivative of \(\tilde{\eta}_0(\phi)\).

\(u_0\) contains a secular term, \(\omega' t\), and, thus, is not bounded. However, the total energy of the system is preserved (Longuet-Higgins 1968b; Vallis 2006), such that

\[
\frac{d}{dt} \iint (\varepsilon u^2 + v^2 + \eta^2) \cos \phi d\lambda d\phi = 0.
\]  

(19)

The assumptions that lead to Eq. (19) are the standard boundary conditions of vanishing meridional velocity at channel walls and periodic boundary conditions in the zonal direction. Thus, secular terms cannot exist in the exact solution of Eqs. (2,6). The total energy for each order of approximation must be preserved. For example, in the zeroth order, the energy of the system is \(E_0 = \iint (v_0^2 + \eta_0^2) \cos \phi d\lambda d\phi\), and is independent of time since \(v_0\) and \(\eta_0\) are simple harmonic (sinusoidal) functions. In Appendix B we show that energy is observed in the first order as well. While secular terms exist in the zero order zonal velocity, they appear only in the first order approximation for the free surface and meridional velocity (Appendix B). The secular term in \(u_0\) limits the validity of the approximated \(u_0\) to times that satisfy \(\omega' t \sim 1\) [see Eq. (17)] while \(v_0\) and \(\eta_0\) are valid for longer times. Thus, for high latitudes where \(\omega'\) is small, the approximation is valid for long times while for low latitudes the approximation is valid for shorter times. Since the frequency \(\omega\) depends on the wave number, for larger wave number (shorter wave length) the validity holds for shorter times. For very long waves for the parameters value used in the numerical simulation described...
below, the approximation is valid for tens of years while for short waves it is valid for years and even shorter time scales.

The dimensional zonal phase speed (in units of length over time) is obtained by multiplying $\omega/k_\lambda$ by $a \cos \phi$:

$$ C(\phi) = -\frac{g H a \cos \phi}{2\Omega a^2 \sin^2 \phi} + \frac{g H k_\lambda^2}{2\Omega \cos^2 \phi}. $$

(20)

For long planetary waves, $k_\lambda$ is very small, so that the second term in the denominator is much smaller than the first term. Thus the phase speed may be approximated as

$$ C(\phi) \approx -\frac{g H \cos \phi}{2\Omega a \sin^2 \phi}. $$

(21)

The zonal periodic boundary conditions impose a trivial quantization condition on the zonal wavenumber, $k_\lambda = \text{integer}$.

The range of latitudes, for which the approximation is valid, depends on the choice of parameters. For the simulations presented below, this range is $|\phi| \gtrsim 20^\circ$.

4. Numerical results

We use the MITgcm to verify the accuracy of the above suggested approximation. Similar to the simulations described in the Appendix A, we consider a flat channel of depth $H = 500$ m on the sphere, extending from $80^\circ$S to $80^\circ$N; the gravitation constant is $g = 0.01$ m/s$^2$. We refrain from extending the channel up to the poles to avoid numerical problems. The simulation is initiated with a spatially periodic sea surface height, with ten sinusoidal cycles in the zonal direction (i.e., a zonal wave length of $36^\circ$), half a sinusoidal cycle in the meridional direction between $60^\circ$S to $20^\circ$S and a full cycle between $20^\circ$N to $60^\circ$N; the
initial free surface is a multiplication of these functions with an amplitude of $A = 4$ m. Mathematically, the initial free surface is $\eta(t = 0) = 4 \sin(10\lambda)\bar{\eta}(\phi)$ where $\bar{\eta}(\phi) = \sin[4.5(\phi - \phi_0)]$ ($\phi$ is given in radians) for $\phi = 60^\circ$S to $20^\circ$S and $\phi_0 = 60^\circ$S and $\bar{\eta}(\phi) = \sin[9(\phi - \phi_0)]$ for $\phi = 20^\circ$N to $60^\circ$N and $\phi_0 = 20^\circ$N; the free surface is zero otherwise. The same $\bar{\eta}(\phi)$ was used to plot the analytical zero order approximations given in Eqs. (17), (18). The initial velocities are set to zero. We choose the initial state to be far from the channel walls to demonstrate that even for internal perturbation and basically arbitrary meridional structure the suggested approximation is valid; other perturbations extending to the channel walls yielded similar agreement between the numerical results and the analytical approximation. We excluded the equatorial region from the initial state since the approximation in its present form is invalid there.

A comparison between the numerical simulations and the approximation, after 15 years of simulations, is shown in Fig. 1. The initial state has propagated to the west where the equatorward parts propagated much faster than the poleward parts. The agreement between the two is good even for this relatively long simulation, indicating the quality of the approximation. Still, the zonal velocity, $u$, exhibits larger difference between the numerical and approximated results, as it contains a secular term, [Eq. (17)]. In addition, there is practically no meridional wave propagation, indicating that in such planetary scales the assumption of meridional free wave is not so accurate.

The zonal phase speed can be estimated using the so called Hovmöller (longitude-time) diagram. The slope of wave crests (or troughs) yields the zonal phase speed at the specified latitude. Fig. 2 depicts the longitude-time diagrams (based on meridional velocity $v$) for latitudes close to the southern edge of the northern wave (Fig. 2a) and close to the northern
edge of the northern wave (Fig. 2b). It is clear that the wave propagates westward approximately at a constant speed and that the zonal phase speed at the southern latitude is much greater than at the northern one.

The numerically estimated zonal phase speed at different latitudes is shown in Fig. 2c. It is based on longitude-time diagrams like the ones shown in Figs. 2a,b calculated at each latitude. The approximated phase speed given in Eq. (20) follows closely the numerical one, where only for latitudes smaller than 30° there is some noticeable difference. The numerical phase speed is approximated based on the mean of the first ten years of simulations; when considering only the first one or two years of simulation, the difference between the numerical and analytical phase speeds is almost indistinguishable. In addition, when considering longer time period for the calculation of the mean phase speed, the difference between the numerical and analytical phase speeds becomes larger, especially for the low latitudes. When considering longer waves in the zonal direction (e.g., $k_\lambda = 2$ instead of $k_\lambda = 10$), the agreement between the numerical and analytical results becomes better and valid for longer times. This observation is not surprising as $\omega'(\phi)$ is large for large wave number $k_\lambda$ and since the approximation is valid for times $t < 1/\omega'$.

As mentioned above, the total energy of the LSWE is conserved [Eq. (19)]. It follows from the nature of the perturbation expansion that the energy of the system should be preserved for each order of approximation as for the zero and first order approximation (Appendix A). The total energy of the numerical solution is not conserved, due to the presence of the viscosity terms. Moreover, shorter waves decay faster. Thus, although the time for which the approximation holds is shorter for shorter waves, it might be sufficient as anyhow in reality (and numerical simulations) shorter waves decay faster. In general, it
is not expected that a perturbation will persist for more than a few decades, because: (i) viscosity acts in reducing the signal amplitude and (ii) other perturbations are likely to occur and overcome the previous perturbation. We thus conclude that from a practical point of view, the time scale over which the suggested approximated is valid is sufficient to describe natural perturbations, at least for the typical parameters used here.

We repeated the above numerical experiments starting with different initial conditions, using different wave numbers in the zonal and meridional directions and using initial zonal or meridional velocities instead of the free surface and obtained similar agreement between the numerical and analytical results. Simulation without the nonlinear advection terms and viscosity terms yielded very similar results with regard to zonal phase speed and wave pattern, so that our numerical results should not be considered as either viscous or nonlinear. Moreover, calculations that are based on the β-plane approximation on a mid latitude channel yielded a phase speed that is very close to the phase speed simulated in the spherical channel where both are very close to the theoretical zonal phase speeds.

5. Discussion and summary

The most widely quoted theory for planetary Rossby waves is based on the quasi-geostrophic approximation. It is, however, known that this theory is valid for meridional scales of up to a few degrees, depending on the Rossby radius of deformation (e.g., Poulin 2009). In Appendix A, this known fact was demonstrated numerically using the MITgcm, starting from a quasi-geostrophic eigenmode, for which the numerical simulation exhibited different zonal wave speeds as a function of latitude. It was also shown that, starting from
the exact eigenmode of planetary Rossby waves on the $\beta$-plane (the parabolic cylinder functions), the structure of the wave is preserved with time and a single zonal phase speed may be associated with each eigenmode. A linear combination of these exact modes yields a wave propagation that is a function of latitude, since the higher modes extend farther poleward where their phase speed is slower. The zonal phase speed follows the classical long Rossby waves dispersion relation.

The exact solutions of off equatorial Rossby waves on the $\beta$-plane—parabolic cylinder functions (e.g., Paldor and Sigalov 2008)—can only be computed numerically to satisfy the boundary conditions. The situation is even more troublesome when dealing with the LSWE on a sphere where an infinite sum of analytical functions (spherical harmonics) (e.g. Longuet-Higgins 1968b) is needed to construct the exact solution. Such a situation calls for the development of approximate solutions to avoid the need for summing up an infinite series or resorting to numerical solutions only.

We proposed a new approximation based on a carefully selected scaling, in which the meridional coordinate appears as a parameter, i.e., the underlying differential equation does not contain derivatives with respect to the meridional coordinate. This equation leads to a wave frequency (and hence phase speed) that explicitly depends on latitude (that plays the role of a parameter), where wave propagation is faster toward the equator. The analytical approximation accounts for arbitrary initial conditions in the meridional direction that satisfy the boundary conditions; the analytical approximation exhibits close similarity with the numerical simulations. The meridional structure of the wave is determined according to the initial conditions and there is no need to find the meridional eigenmodes to find the time development of the initial perturbation, at least up to some finite times during which
the scaling remains valid. The zonal velocity in the approximation contains a secular term that grows with time and, thus, the approximation is valid for times \( t \approx 1/\omega' \) where \( \omega \) is the wave frequency. Multiple time scale technique is often used to eliminate secular terms in perturbation expansion, by effectively updating the frequency of the lower order approximation. We have found that this technique is not applicable in our case.

The LSWE have only a single small parameter, \( \varepsilon \). When applying the scaling of the different variables in the LSWE the outcome is that \( \varepsilon \) multiplies only the time derivative of the zonal momentum equation, putting the meridional velocity in geostrophic balance. This is a consequence of the different scaling used for the zonal and meridional coordinates. The use of different scaling for the zonal and meridional directions is not new and has been used, for example, in the study of equatorial wave dynamics (e.g., Cane and Sarachik 1976, 1981; Emile-Geay and Cane 2009), where the time derivative in the meridional momentum equation is multiplied by the small parameter and hence, to a zero order approximation, the zonal velocity is in geostrophic balance. Although in this case the corresponding differential equation does contain derivatives with respect to the meridional coordinate, the solution is relatively simple.

The major outcome of the approximation proposed here is that the wave frequency is a function of the meridional coordinate. This leads to a secular term in the zonal velocity, through the meridional derivative of the free surface \( \eta \). Such a behavior is not new: Under the long wave assumption the planetary geostrophy may be obtained (e.g., Cessi and Louazel 2001; Primeau 2002; Vallis 2006) and when linearized, the time derivatives in both momentum equations may be neglected, to a zero order approximation. This leads to a differential equation without derivatives with respect to the meridional coordinate, but with the explicit
meridional dependence of the Coriolis parameter (e.g., Meyers 1979; Emile-Geay and Cane 2009). Thus, as for the approximation suggested here, the wave frequency and zonal phase speed in these studies also depend on latitude and hence the zonal velocity contains a secular term. Yet, the solution obtained by the planetary geostrophic approximation is less general than the one presented above, as more terms are neglected from the LSWE. In addition, we also provide the next order approximation (Appendix B).

It was shown previously that when changes in bottom topography are very moderate with respect to the changes in the wave length, it is possible to replace the mean water depth by the local water depth (e.g., Pedlosky 2003). It is possible to adopt a similar approach for change in the Coriolis parameter, meaning that when the meridional wave length is very short, it is possible to replace the mid-channel Coriolis parameters by the local ones. The approximation proposed here is applicable to short and long zonal wave length (as long it is significantly shorter then the meridional wave length). In addition, wave frequency as a function of the meridional coordinate was assumed to estimate the equatorial wave guide width (e.g., Gill 1982; Pedlosky 2003).

The classical quasi-geostrophic Rossby waves zonal phase speed for long planetary waves is (Gill 1982; Pedlosky 1987) $C_x \approx -\beta g H / f_0^2$. Our approximation for the $\beta$-plane long Rossby wave zonal phase speed is reproduced, by replacing the local Coriolis parameter $f_0$ with $f_0 + \beta y$. Moreover, it is possible to reproduce Eq. (21) by replacing the $\beta$-plane parameters, $f_0$ and $\beta$, with their spherical counterparts $2\Omega \sin \phi$ and $(2\Omega / a) \cos \phi$. However, this correspondence does not mean that the classical Rossby wave theory predicts that a single specific planetary wave will propagate with different speeds at different latitudes. In fact, the present theory bolsters the applicability of the results of the quasi-geostrophic
theory regarding the phase speed but without making any assumptions on the meridional variation of the amplitude of the dependent variables, $u$, $v$ and $\eta$.

It is interesting to compare the dispersion relation obtained above in mid-latitudes to those of equatorial Rossby waves on the $\beta$-plane, $\omega = -\beta k / [k^2 + \beta(2n + 1)/\sqrt{gH}]$, where $k$ is the zonal wave number and $n$ is an integer associated with the Hermite polynomials of the meridional direction. For low $n$ and relatively large $k$ (short zonal wave length), the dispersion relation reduces to $\omega = -\beta/k$ which is what we get when assigning $\phi = 0$ in the dispersion relation we obtained above [Eq. 12]. In addition, for extra-tropical Rossby waves on the $\beta$-plane, the dispersion relation obtained when using $\omega$ that is independent of $\phi$ (e.g., Paldor and Sigalov 2008), there is an additional term in the denominator associated with the meridional eigenstate, a term which is small for low meridional modes. In the dispersion relation given above [Eq. (12)] the term that is associated with the meridional eigenstate is absent. Thus, our suggested approximation may be regarded as the long-meridional wave approximation. In the case of long waves both for the zonal and meridional directions, our approximation is still valid and in essence relaxes to the planetary geostrophy solution. In addition, the dispersion relation we obtained may be regarded as the upper limit of the frequency $\omega$, as additional positive terms (associated with the meridional eigenstate) in the denominator of the dispersion relation reduce the frequency.

The standard mathematical set-up for the study of the properties of Rossby waves is a channel (e.g., Pedlosky 1987; Gill 1982). Obviously, channel-like configurations do not exist on planetary scales. Thus, the channel set-up, required by the traditional mathematical approach was questioned in the past. In the approximation suggested here the “channel” can be extended from pole-pole (which natural for global scale waves) while an internal basin
off-equatorial perturbation may be successfully simulated. This is due to the fact that the meridional coordinate acts like a parameter in the suggested approximation, enabling the use of an arbitrary meridional structure provided only that it satisfies the boundary conditions, which obviously includes internal basin perturbation. We thus conclude that the channel configuration is not a real issue in the suggested approximation.

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APPENDIX A

Rossby waves in a mid latitude channel:

quasi-geostrophic approximation versus numerical simulations

To study the characteristics of the first baroclinic mode of planetary Rossby waves, we numerically solve the primitive equations on the $\beta$-plane. We consider a channel extending from $20^\circ$N to $50^\circ$N with constant depth of $H=500$ m and gravitation constant of $g=0.01$ ms$^{-2}$, values that roughly represent the observed oceanic mean values for the first baroclinic mode. The zonal extent is $360^\circ$. Water density is assumed to be constant. Hence, only the non-linear horizontal momentum equations and the continuity equation are taken into account. These equations are numerically solved by the MITgcm (Adcroft et al. 2002; Marshall et al. 1997a, 1997b) subject to the boundary conditions of free slip for the zonal velocity and zero meridional velocity at channel walls. Periodic boundary conditions are assumed in the zonal direction. The resolution is $1^\circ \times 0.5^\circ$ for the zonal respectively, and meridional directions and one level in the vertical direction.

The simulation is initiated with a spatially periodic sea surface height and velocities, with two sinusoidal cycles in the zonal direction and half a sinusoidal cycle in the meridional direction; see Fig. 3a. The initial state is chosen to be an eigenmode of the quasi-geostrophic
approximation for Rossby waves. After 20 years of simulation the wave propagates westward, much farther at low latitudes than at high latitudes, as can be seen in Fig. 3b, in which the free surface height, $\eta$, is depicted. The zonal and meridional velocities exhibit similar latitudinal dependence.

The westward propagation of the wave, the main property that characterizes Rossby waves, is evident from Fig. 3. Moreover, the analyzed phase speed for each latitude (not shown) coincides with the well known long Rossby wave phase speed (e.g., Pedlosky 1987) as a function of latitude:

$$C_x(y) \approx -\frac{\beta g H}{(f_0 + \beta y)^2},$$  \hspace{1cm} (A1)

where $f_0$ and $\beta$ are the $\beta$-plane parameters; Eq. (A1) may be obtained by replacing the $f_0$ with $f_0 + \beta y$ in the known phase speed $C_x(y) = -\beta g H / f_0^2$. However, according to the quasi-geostrophic approximation, the wave should have propagated uniformly with latitude with a phase speed corresponding to the middle of the channel, $C_x(y_0)$. The difference in the zonal phase speed between the northern part of the channel and the southern part is large and significant. The differences are noticeable even over a few degrees latitude (see below). It is thus clear that under the typical parameter values we use here, the quasi-geostrophic Rossby wave eigenmodes do not accurately approximate the exact eigenmodes of the system.

We repeated the above experiment for different channel widths, corresponding to meridional extents of 20° to 35°N (Fig. 3c) and 20° to 27.5°N (Fig. 3d); the meridional resolution is 0.25° and 0.125° respectively. For both cases the wave propagation dependence on latitude is large. The quasi-geostrophic approximation is more accurate for channels located further poleward. We use a sufficiently small viscosity coefficient to avoid a smearing effect is the
meridional direction, which may lead to seemingly uniform propagation of the wave.

It is possible to derive Rossby waves on the \( \beta \)-plane in a channel, by assuming a solution that is a product of a traveling wave solution in the zonal direction and unknown function that satisfies the boundary conditions in the meridional direction. Then, an ordinary differential equation is obtained for the meridional velocity where the independent variable is the meridional coordinate, \( y \). The general solution for this differential equation is the parabolic cylinder function (e.g., Gill 1982; Paldor and Sigalov 2008; Poulin 2009). The parabolic cylinder functions have an oscillatory region equatorward and exponentially decaying (or growing) region poleward. The wave frequency, and hence the dispersion relation, is determined according to the boundary conditions. The first four eigenmodes (numbered “0” to “3”) are depicted in Fig. 4a; as the mode number increases, the decay occurs at higher latitudes and more oscillations are present at the lower latitudes. The number of extremal points is equal to the mode number plus one. For sufficiently large mode numbers the oscillations extend to the northern wall of the channel.

We next check whether the latitudinal dependence observed for the quasi-geostrophic initial conditions also exists in the exact solution of Rossby waves on the \( \beta \)-plane. For this purpose we run the same simulation as in Fig. 3 but now starting with the parabolic cylinder function eigenmodes for the meridional velocity instead of the sinusoidal functions; the initial free surface height and the zonal velocity are calculated based on the initial meridional velocity. The initial state and the wave pattern after 20 years of simulations are presented in Fig. 5; here we use as an example mode “3” and a linear combination of the first four parabolic cylinder modes. While the structure of mode “3” remained almost unchanged through its traveling westward, the linear combination of the first four modes
yielded a faster zonal wave speed toward the equator. These numerical simulations validate the exact parabolic cylinder function solution.

We numerically calculate the zonal phase speed as a function of latitude for each of the first four parabolic cylinder eigenmodes and of their linear combination. The results are depicted in Fig. 4b. Also included are the theoretical phase speeds of the modes (horizontal dashed lines) and the classical long Rossby wave zonal phase speed [Eq. (A1), heavy dotted line]. The zonal phase speed is almost constant over the part that the wave is practically non-zero and coincides with the theoretical parabolic cylinder phase speed. These simulations demonstrate in fact that the parabolic cylinder function is the exact solution to the problem.

The initial state of the linear combination of the first four parabolic cylinder modes propagated with different zonal phase speeds for different latitudes (Figs. 4 and 5), similar to the simulation of quasi-geostrophic eigenmode initial conditions depicted in Fig. 3. Moreover, the numerically calculated phase speed follows the classical zonal phase speed [Eq. (A1)]. It is fairly easy to understand the different phase speeds for different latitudes since higher parabolic cylinder modes reach higher latitudes and propagate more slowly, and hence the reason for capturing slower phase speed poleward. However, the good agreement with Eq. (A1) is more difficult to understand; see similar observation in Fig. 12.3 of Gill (1982), obtained for the discrete equatorial modes.

The vertical dotted lines in Fig. 4 indicate the crossing points between the parabolic cylinder modes phase speeds and the long Rossby waves phase speed [Eq. (A1)]. These crossing points, also known as the “turning latitude” points, occur at the transition between the oscillatory and the exponentially decaying parts of the parabolic cylinder functions. Recent studies (Paldor et al. 2007; Poulin 2009) discussed the latitude that should be associated
with the Rossby wave eigenmode, in effort to explain the greater observed mid-latitudes zonal phase speed (Chelton and Schlax 1996), as compared with the classical Rossby wave phase speed. Our numerical simulations suggest that (i) the exact Rossby wave phase speed follows very closely the classical zonal phase speed [Eq. (A1)] and that (ii) the latitude that separates between the oscillatory and exponentially decaying parts of the parabolic cylinder eigenmode can characterize the eigenmode.

It is not straightforward to numerically calculate the eigenvalues of the parabolic cylinder eigenmodes—these eigenvalues cannot be found analytically. Moreover, when considering the more complete problem of Rossby waves on the sphere, the exact solution is even more complicated as it is expressed as an infinite sum of spherical harmonics (Longuet-Higgins 1968a). The approximation suggested in the text aims in providing a simple approximation for Rossby waves on the sphere.
Higher order approximation

The treatment presented below uses non-dimensional variables. It is possible to expand Eq. (6) in terms of \( \varepsilon \) as follow:

\[
(L_0 + \varepsilon L_1 + \varepsilon^2 L_2)\eta = 0 \tag{B1}
\]

where

\[
L_0 = \sin^2 \phi \cos \phi (\sin^2 \phi \partial_t - \partial_\lambda) - \frac{\sin^2 \phi}{\cos \phi} \partial_{\lambda t}
\]

\[
L_1 = \sin \phi (\cos^2 \phi + 1) \partial_t \phi - \cos \phi \sin^2 \phi \partial_{t^3 \phi} + \cos \phi \partial_{t^2 \lambda} + 2 \cos \phi \sin^2 \phi \partial_t^3 - \frac{1}{\cos \phi} \partial_t^3 \partial_\lambda^2 \tag{B2}
\]

\[
L_2 = \sin \phi \partial_t^5 \phi - \cos \phi \partial_t^3 \partial_\phi^2 + \cos \phi \partial_t^5.
\]

Using these operators, the equation for the \( n \)th order approximation in \( \varepsilon \) is:

\[
L_0 \eta_n + L_1 \eta_{n-1} + L_2 \eta_{n-2} = 0, \tag{B3}
\]

where we set \( \eta_{-1} = \eta_{-2} = 0 \) such that this equation will account also for the zero and first order approximations. The periodic boundary conditions in the zonal direction impose a solution of the form:

\[
\eta(\lambda, t, \phi) = e^{ik\lambda} \tilde{\eta}(t, \phi). \tag{B4}
\]
Using Eq. (B4) and introducing the expression for \( \omega \) [Eq. (11)], \( L_{0,1,2} \) are found to be all proportional (with the same proportionality coefficient) to:

\[
M_0 = i\omega + \partial_t \\
M_1 = -\frac{\omega}{k_\lambda} [(2 \cot \phi + \tan \phi) \partial_\phi - \partial_t \phi] + \frac{ik_\lambda}{\sin^2 \phi} \partial_t + 2(1 + 2k_\lambda^2 / \sin^2 2\phi) \partial_t^3 \\
M_2 = -\frac{\omega}{k_\lambda} \sin \phi \partial_t^3 \partial_\phi - \partial_t^3 \partial_\phi + \partial_t^5 \\
\]

(B5)

The equation for \( \eta \) is \((M_0 + \varepsilon M_1 + \varepsilon^2 M_2)\eta = 0 \) and the \( n \)th order approximation equation is \( M_0 \eta_n + M_1 \eta_{n-1} + M_2 \eta_{n-2} = 0 \). Note that \( M_0 \) contains only the first time derivative, a fact that helps in finding the first (and higher) order approximation. In addition, in these equation, the meridional coordinate acts like a parameter and only a first order ordinary differential equation has to be solved for each order of approximation.

Using \( M_0 \) and \( M_1 \) operators it is possible to find first order approximation for \( \eta \) using

\[
M_0 \eta_1 + M_1 \eta_0 = 0, \tag{B6}
\]

where \( \eta_0 = \tilde{\eta}_0(\phi) \exp i(k_\lambda \lambda - \omega t) = \tilde{\eta}_0(\phi) \exp (i\theta) \) and \( \eta_1 = f_1(t, \phi) \exp (ik_\lambda \lambda) \). The aim is to find a particular solution to the above equation. The \( \phi \) derivatives in \( M_1 \) lead to terms that are proportional to \( t \) and \( t^2 \), as \( \omega \) is a function of \( \phi \). The equation to solve is:

\[
 i\omega \eta_1 + \partial_t \eta_1 + (ia_0 + a_1 t + ia_2 t^2)e^{i(k_\lambda \lambda - \omega t)} = 0, \tag{B7}
\]

where the coefficients \( a_j(\phi) \) are given by:

\[
a_0(\phi) = -\frac{\omega}{k_\lambda} \left\{ -k_\lambda \csc^2 \phi \omega^2 \tilde{\eta}_0 + 2(1 + 2k_\lambda^2 \csc^2 \phi) \omega^2 \tilde{\eta}_0 + 2\omega \tilde{\eta}_0' \right. \\
+ \tilde{\eta}_0[\omega'' - (2 \cot \phi + \tan \phi) \omega'] + \omega[\tilde{\eta}_0'' - (2 \cot \phi + \tan \phi) \tilde{\eta}_0''], \tag{B8}
\]

\[
a_1(\phi) = -\frac{\omega}{k_\lambda} \left( 2\tilde{\eta}_0 \omega' + \omega[2\omega' \tilde{\eta}_0' + \tilde{\eta}_0(\omega'' - (2 \cot \phi + \tan \phi) \omega')] \right), \tag{B9}
\]

\[
a_2(\phi) = \omega^2 \omega'^2 \tilde{\eta}_0 / k_\lambda. \tag{B10}
\]
The particular solution to the above equation is:

\[ \eta_1 = -(ia_0 t + a_1 t^2 / 2 + ia_2 t^3 / 3)e^{i\theta}. \]  

(B11)

By taking the imaginary (or real) value of \( \eta_1 \) we obtain the real value solution

\[ \eta_1 = -a_0 t \cos \theta - \frac{1}{2} a_1 t^2 \sin \theta + \frac{1}{3} a_2 t^3 \cos \theta. \]  

(B12)

The first order approximation zonal and meridional velocities \( u_1 \) and \( v_1 \) may be found using Eqs. (13).

It is now left to verify whether the first order approximation solution satisfies the energy preservation given in Eq. (19). To a first order in \( \varepsilon \), Eq. (19) may be written as follows

\[ E_1 = \iiint (u_0^2 + 2v_0 v_1 + 2\eta_0 \eta_1) \cos \phi d\lambda d\phi, \]  

(B13)

where \( E_1 \) stands for the first order approximation for the total energy of the system. The terms that contain multiplication of the form \( \cos \theta \sin \theta \) become zero when integrating with respect to \( \lambda \), due to the periodic boundary conditions in the zonal direction. In addition, terms that are independent of \( t \) after the integration are not of interest as they are obviously conserved with time. After excluding these terms and integrating with respect to \( \lambda \), \( E_1 \) becomes:

\[ E_1 = -\pi t^2 \int_{\phi_1}^{\phi_2} \left[ \omega' \eta_0 + 2\omega' \eta_0' + \omega^2 \eta_0 - \omega' (2 \cot \phi + \tan \phi) \eta_0 \right] \frac{\eta_0 \cos \phi}{\sin^2 \phi} d\phi, \]  

(B14)

where \( \phi_{1,2} \) are the locations of the channel walls. It is easy to show that:

\[ E_1 = -\pi t^2 \left[ \frac{\omega' \cos \phi \eta_0^2}{\sin^2 \phi} \right]_{\phi_1}^{\phi_2} = 0, \]  

(B15)

as \( \eta_0 \) vanishes at the channel walls, \( \phi_{1,2} \), since \( v_0 \) vanishes at the channel walls and \( v_0 \) is proportional to \( \eta_0 \) [Eq. (16)].
It is expected that the energy of the system will be preserved for each order of approximation in $\varepsilon$, even in the presence of secular terms.
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