STABILITY OF SECOND-ORDER ASYMMETRIC LINEAR MECHANICAL SYSTEMS WITH APPLICATION TO ROBOT GRASPING

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ABSTRACT

This technical correspondence presents a surprisingly simple analytical criterion for the stability of general second-order asymmetric linear systems. The criterion is based on the fact that if a symmetric system is stable, adding a small amount of asymmetry would not cause instability. We compute analytically an upper bound on the allowed asymmetry such that the overall linear system is stable. This stability criterion is then applied to robot grasping arrangements which, due to physical effects at the contacts, are asymmetric mechanical systems. We present an application of the stability criterion to a 2D grasp arrangement.

INTRODUCTION

This technical correspondence is concerned with the stability of second-order linear systems that have an asymmetric stiffness matrix. Our goal is to provide an analytical criterion for the stability of systems of the form:

$$M\ddot{p} + K_d\dot{p} + K_p p = 0, \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ and $K_d \in \mathbb{R}^{n\times n}$ are symmetric positive definite, and $K_p \in \mathbb{R}^{n\times n}$ is *asymmetric*. Such systems arise in the linearized dynamics of robot grasping arrangements [9], and in other applications such as feedback control. See, for instance, [7] and [5, p. 36].

Researchers have taken the following approach to the investigation of general asymmetric systems, where M, K_d, and K_p are asymmetric. Their approach is based on transforming the asymmetric system into a symmetric one. The subclass of asymmetric systems that can be transformed into symmetric systems is called symmetrizable systems. Inman has introduced necessary and sufficient conditions for a subclass of such systems to be symmetrizable via similarity transformation [4]. Ahmadian and Chou have developed a systematic technique for computing the coordinate system in which the symmetrizable system is symmetric [2]. Coghey and Ma have given a condition for transforming the system into a decoupled diagonal system [3]. Utilizing equivalence transformation rather than similarity transformation enables the subclass of symmetrizable systems to be enlarged [1, 8]. All these results are exact and give conditions for the stability of the original asymmetric system. However, only subclasses of asymmetric systems can be treated in these ways, and the application of stability criteria based on transformation to symmetric systems is cumbersome.

In this technical correspondence we develop a simple criterion for the stability of asymmetric systems of the form (1). In the context of robot grasping applications, this stability criterion leads to a synthesis rule that indicates which contact points and what preloading profile guarantee stable grasp.

We make the following two assumptions, which are motivated by consideration of the grasping application. First, as in many mechanical systems, we assume that the inertia and damping matrices, M and K_d, are symmetric positive definite matrices. Second, we assume that the symmetric part of the stiffness matrix, $(K_p)_s = \frac{1}{2}(K_p + K_p^T)$, is positive definite. This assumption has been shown to hold true for almost every robot grasping application [9].

The stability criterion is based on the idea that if the symmetric system is asymptotically stable, one can add a bounded amount of asymmetry and the system will remain stable. In our solution we compute an upper bound on the norm of $(K_p)_{as} = \frac{1}{2}(K_p - K_p^T)$ such that the eigenvalues of the first-order equation recast from equation (1) are located in the open left half plane. After establishing the stability criterion for such systems, we illustrate the applicability of the result for analyzing the stability of robot grasping arrangements.

STABILITY OF 2$^{\text{nd}}$-ORDER ASYMMETRIC SYSTEMS

For simplicity we begin with the following system:

$$\ddot{p} + K_d \dot{p} + K_p p = 0, \tag{2}$$

which is identical to (1), except that here M is the identity matrix. The following theorem states that if the skew-symmetric part of K_p, $(K_p)_{as}$, is sufficiently small, the system (2) is globally asymptotically stable.

Theorem 1 (global asymptotic stability). *Consider the system (2). Let $\beta \in \mathbb{R}$ be the minimal eigenvalue of K_d. Let $\alpha \in \mathbb{R}$ be the minimal eigenvalue of $(K_p)_s$, and let $\gamma \in \mathbb{R}$ be the matrix norm[1] of the skew-symmetric part of K_p. If*

$$|\gamma| < \sqrt{\alpha}\beta,$$

the system (2) is globally asymptotically stable.

Proof. The system (2) can be written as

$$\frac{d}{dt}\begin{pmatrix} p \\ \dot{p} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix}}_{A} \begin{pmatrix} p \\ \dot{p} \end{pmatrix}$$

For global asymptotic stability, it suffices to show that the real part of the eigenvalues of A is negative. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A with corresponding non-zero eigenvector $v = (v_1, v_2) \in \mathbb{C}^{2n}$. Note that each v_i is a complex vector in $\mathbb{C}^n$. Then

$$\begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ -K_p v_1 - K_d v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Since $(K_p)_s > 0$ K_p is non-singular. This implies that $\lambda = 0$ cannot be an eigenvalue of A. Since $\lambda \neq 0$, it follows that $v_1 \neq \vec{0}$ and $v_2 \neq \vec{0}$. Hence, we may assume without loss of generality that $v_1^* \cdot v_1 = 1$, where $*$ denotes complex conjugate transpose. Based on this choice, we can write $\lambda^2 = v_1^* \lambda^2 v_1 = v_1^* \lambda v_2 = v_1^*(-K_p v_1 - K_d v_2) = -v_1^* K_p v_1 - \lambda v_1^* K_d v_1$, where we used the relations $\lambda v_1 = v_2$ and $\lambda v_2 = -K_p v_1 - K_d v_2$. Since $K_d > 0$, the scalar $\tilde{\beta} = v_1^* K_d v_1$ is positive real. Similarly, the scalar $\tilde{\alpha} = v_1^*(K_p)_s v_1$ is also positive real. Since $(K_p)_{as}$ is skew-symmetric, we can write $j\tilde{\gamma} = v_1^*(K_p)_{as} v_1$, where $j = \sqrt{-1}$ and $\tilde{\gamma}$ is real. Substituting these scalars into the quadratic equation in λ gives

$$\lambda^2 + \tilde{\beta}\lambda + \tilde{\alpha} + j\tilde{\gamma} = 0. \tag{3}$$

Note that every eigenvalue of A satisfies this equation. The solution of (3) is:

$$\lambda_{1,2} = \frac{1}{2}\left(-\tilde{\beta} \pm \sqrt{\tilde{\beta}^2 - 4(\tilde{\alpha} + j\tilde{\gamma})}\right). \tag{4}$$

[1]The matrix norm is defined as $||E|| = max\{||Eu||\}$ over all vectors $||u|| \leq 1$.

Let us pause to recall how one computes the square root of a complex number. Consider a complex number $z = a + jb$ with a norm $|z| = \sqrt{a^2+b^2}$ and argument $\theta = \arctan(b/a)$. Then $\sqrt{z} = \pm(a^2+b^2)^{\frac{1}{4}}\angle\frac{\theta}{2}$, and in cartesian coordinates $\sqrt{z} = \pm(a^2+b^2)^{\frac{1}{4}}\left(\cos\left(\frac{\theta}{2}\right) + j\sin\left(\frac{\theta}{2}\right)\right)$. Since $\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}}$, we use the trigonometric identity $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos(\theta)}{2}}$ to obtain

$$\mathsf{Re}\{\sqrt{z}\} = \pm\frac{(a^2+b^2)^{\frac{1}{4}}}{\sqrt{2}}\left(1+\frac{a}{\sqrt{a^2+b^2}}\right)^{\frac{1}{2}}.$$

In our case $a = \tilde{\beta}^2 - 4\tilde{\alpha}$ and $b = -4\tilde{\gamma}$, and (4) implies that

$$\mathsf{Re}\{\lambda_{1,2}\} = -\frac{\tilde{\beta}}{2} \pm \frac{\left((\tilde{\beta}^2-4\tilde{\alpha})^2+16\tilde{\gamma}^2\right)^{\frac{1}{4}}}{2\sqrt{2}}\left(1+\frac{(\tilde{\beta}^2-4\tilde{\alpha})}{\sqrt{(\tilde{\beta}^2-4\tilde{\alpha})^2+16\tilde{\gamma}^2}}\right)^{\frac{1}{2}}$$

The requirement $\mathsf{Re}\{\lambda_{1,2}\} < 0$ introduces an inequality in $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$. Rearranging terms in this inequality gives the equivalent inequality,

$$\left(4\tilde{\alpha}+\tilde{\beta}^2\right)^2 > \left(\tilde{\beta}^2-4\tilde{\alpha}\right)^2 + 16\tilde{\gamma}^2.$$

Cancelling similar terms yields the inequality

$$|\tilde{\gamma}| < \sqrt{\tilde{\alpha}}\tilde{\beta}. \tag{5}$$

For stability we must ensure that (5) holds for every $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$. In other words, (5) must hold for every eigenvalue λ and every associated eigenvector v of A. Therefore we bound $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ as follows. First, $0 < \alpha = \lambda_{min}\left((K_p)_s\right) \le v_1^*(K_p)_s v_1 = \tilde{\alpha}$. Second, $0 < \beta = \lambda_{min}(K_d) \le v_1^* K_d v_1 = \tilde{\beta}$. Third, $|\gamma| = ||(K_p)_{as}|| \ge |v_1^*(K_p)_{as}v_1| = |j\tilde{\gamma}| = |\tilde{\gamma}|$. Using these bounds, $\gamma < \sqrt{\alpha}\beta$ implies that $|\tilde{\gamma}| < \sqrt{\tilde{\alpha}}\tilde{\beta}$ for every $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$. □

Note that the theorem gives only sufficient stability condition, and the proof does not indicate what should be the necessary condition for global asymptotic stability. Next, we present a corollary that adapts the theorem to a global asymptotic stability criterion for systems that contain a non-unit inertia matrix.

Corollary 2.1. *Consider the following system*

$$M\ddot{p} + K_d\dot{p} + K_p p = 0, \tag{6}$$

where all parameters are as above, except for the matrix M which is symmetric positive definite. Let $\beta > 0$ be the minimal eigenvalue of $M^{-1/2}K_dM^{-1/2}$. Let $\alpha > 0$ be the minimal eigenvalue of $M^{-1/2}(K_p)_sM^{-1/2}$, and let $\gamma \in \mathbb{R}$ be the matrix norm of $M^{-1/2}(K_p)_{as}M^{-1/2}$. If

$$|\gamma| < \sqrt{\alpha}\beta$$

the system (6) is globally asymptotically stable.

Proof. We define the coordinate transformation $\tilde{p} = M^{1/2}p$ or $p = M^{-1/2}\tilde{p}$. (A similar transformation appeared in [5, p. 87].) Note that the matrices $M^{1/2}$ and $M^{-1/2}$ are symmetric positive definite. Moreover, we have that $M = M^{1/2}M^{1/2}$ and $M^{-1} = M^{-1/2}M^{-1/2}$. Substituting the new coordinates into (1) and premultiplying by $M^{-1/2}$ gives

$$\ddot{\tilde{p}} + \underbrace{M^{-1/2}K_dM^{-1/2}}_{\tilde{K}_d}\dot{\tilde{p}} + \underbrace{M^{-1/2}K_pM^{-1/2}}_{\tilde{K}_p}\tilde{p} = 0.$$

This system is exactly of the form used for theorem 1, but instead of K_d and K_p we now have $\tilde{K}_d$ and $\tilde{K}_p$, respectively. If the latter system is asymptotically stable, it entails that (1) is asymptotically stable, since the two systems differ only by coordinate transformation. The global asymptotic stability of (6) therefore follows from theorem 1. □

We conclude this section with a simple numerical example that shows the applicability of the stability criterion.

Example: Consider the dynamical system

$$\begin{bmatrix} 10 & 0 \\ 0 & 11 \end{bmatrix} \ddot{p} + \begin{bmatrix} 4 & 1 \\ 1 & 5 \end{bmatrix} \dot{p} + \begin{bmatrix} 8 & s \\ -s & 9 \end{bmatrix} p = 0, \tag{7}$$

where s is a free parameter. The matrices M, K_d, and the symmetric part of K_p are all symmetric positive definite. Hence, when $s = 0$ the system is symmetric and asymptotically stable. Qualitatively, increasing the value of s increases the asymmetric part of the stiffness matrix. Calculation of α, β, and γ yields $\alpha = \frac{4}{5}$, $\beta = 0.328$, and $\gamma = \frac{s}{\sqrt{110}}$. Therefore, the stability condition of corollary 2.1 becomes the condition $|s| < 3.078$. For comparison we numerically calculated the eigenvalues of the 4×4 matrix A. It turns out that for $0 \leq s < 3.920$ the system (7) is asymptotically stable (A's eigenvalues are in the left half plane). We can see that apart from being conservative, our stability condition correctly predicts the system's global asymptotic stability.

APPLICATION TO GRASP SYNTHESIS

In this section our objective is to determine the stability of frictional grasps or fixtures. We consider a grasp, or fixture, arrangement where a 2D object $\mathcal{B}$ is held by stationary 2D bodies $\mathcal{A}_1,\ldots,\mathcal{A}_k$ that represent fingertips or fixturing elements. We assume frictional contacts between the stationary bodies $\mathcal{A}_1,\ldots,\mathcal{A}_k$ and $\mathcal{B}$. The usual assumption made in the solid mechanics literature is that the contacting bodies are *quasi-rigid*, which means that their deformations due to compliance effects are localized to the vicinity of the contacts [6]. This assumption is always valid for all bodies that are not made of exceptionally soft material and do not contain slender substructures [10]. The quasi-rigidity assumption allows us to describe the overall motion of $\mathcal{B}$ relative to the stationary bodies $\mathcal{A}_1,...,\mathcal{A}_k$ using rigid body kinematics. Since the grasping bodies are stationary, we focus on $\mathcal{B}$'s *configuration space* (c-space). The c-space of a planar object is parametrized by $q = (d,\theta) \in \mathbb{R}^2 \times \mathbb{R}$, where d is $\mathcal{B}$'s position and θ is a parametrization of $\mathcal{B}$'s orientation.

We have derived the following linearized dynamics of a quasi-rigid object $\mathcal{B}$ held in equilibrium grasp by stationary quasi-rigid bodies $\mathcal{A}_1,\ldots,\mathcal{A}_k$ [9]:

$$M(q_0)\Delta\ddot{q} + K_d(q_0)\Delta\dot{q} + K_p(q_0)\Delta q = 0, \tag{8}$$

where q_0 is the grasped object equilibrium configuration and Δq is the deviation of the actual configuration from the equilibrium.

In grasping applications $M(q_0)$ is the inertia matrix, and $K_d(q_0)$ is the damping matrix. Both matrices are symmetric and positive definite. The matrix $K_p(q_0)$ is the grasp stiffness matrix associated with the mechanics of quasi-rigid frictional contacts. This matrix is composed of the individual contact stiffness matrices, which are asymmetric. See [9] for more details.

The asymmetry of K_p strongly depends on the direction of the contact forces, which in some cases can be selected during grasp synthesis. The magnitude of the matrix norm of $(K_p)_{as}$ increases as the angle between the contact force and the normal at the contacts increases.

For example, consider the two-finger frictional grasp shown in figure 1. The example shows a grasp of a wedge-like object, which has a head angle ϕ and base angle $90°-\phi$ as shown in the figure. Hence, the example is actually a grasp of a family of wedge-like objects with different head angles. In this example we assume that the friction is sufficiently large that the fingers do not slide. Of course, the two-finger grasp forms an equilibrium grasp. However, the stiffness matrix K_p is asymmetric and local deformations at the contacts can cause instability. In the example, if the contact forces F_1 and F_2 are collinear with the normals at the contacts n_1 and n_2, then K_p is symmetric. When the contact forces rotate away from the normal directions the matrix norm $||(K_p)_{as}||$ increases. The rotation of the contact forces with respect to the normal is due to the grasping of different objects with varying ϕ angles. The stability condition of corollary 2.1 places a limit on the amount of asymmetry allowed. Consequently, it bounds the value of the allowed angle ϕ. Computation of the maximal ϕ angle reveals that the grasp is stable for $\phi < 12.68°$.

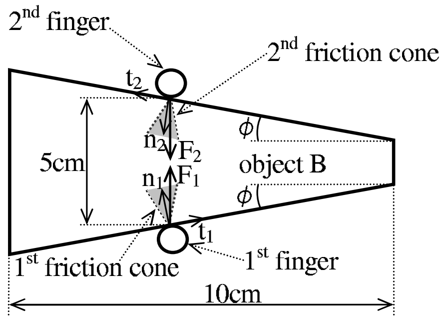

Figure 1. A two-finger grasp of a family of wedge-like objects.

CONCLUSION

Adding an asymmetric matrix to a stable symmetric second-order system has the potential to cause instability. In order to avoid such instability, we establish an analytic bound on the amount of asymmetry that is guaranteed to keep the asymmetric system globally asymptotically stable.

Recent results show that a frictional contact stiffness matrix is asymmetric. As a result, the grasp stiffness matrix of the entire grasp is asymmetric. We obtained a concise condition for the global asymptotic stability of the grasp linearized dynamics, and therefore a local asymptotic stability for the nonlinear system.

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