

Separating Equilibria in Public Auctions

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Abstract

We consider two private-value auctions where the prize in one is higher than the prize in the other. We show that a separating equilibrium exists where bidders with a high valuation attend the auction with the higher prize while the weak bidders attend the auction with the lower prize. In addition, we prove that a weak separating equilibrium exists where the strong bidders attend the high prize auction while the weak bidders randomize and may attend either auction, although with a higher probability of attending the low prize auction.

Keywords: auctions, separating equilibrium, asymmetric information.

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1 Introduction

There are numerous practical situations where authorities offer the public many objects through auctions. Some examples would be, sport competitions (with different leagues), cellular frequencies licenses, broadcasting rights, mineral rights, research grants, construction contracts (such as power plants, parks, dams, public buildings, etc.), among numerous others. These auctions are characterized by nonidentical multiunit prizes. Assuming that the authorities allow for a bidder in the competition to win up to a single prize, an efficient selling mechanism might be possible in some circumstances, but is not always welcome. While efficient outcome is considered as allocation of the prizes to the bidders with the highest valuation, the authorities may also wish to achieve different social goals. For instance, authorities may request a mechanism that will give weak competitors the chance to survive and win some of the prizes. This type of mechanism commonly appears in sport competitions where strong teams compete in a high league while weak teams compete in a lower one. In the case of a public auction for mineral rights or construction contracts, the prizes may differ in size or amount and potential participants may attend an auction with a high or low prize. A well known example of auctions with different prizes can be found in frequency (spectrum) auctions for third generation mobile telephones. In the UK, authorities offered $15 - 15MHz$ (large)¹ licences and $10 - 10MHz$ (small) licences (for details see, Börgers and Dustmann 2003, 2005). In US auctions for nationwide nar-

¹ $15 - 15MHz$ represents a bandwidth of $15MHz$ from the network to the users and $15MHz$ from the users to the network. $15MHz$ means that the bandwidth is only from the network to the user.

rowband frequencies, the FCC offers three different licenses: $50 - 50KHz$, $50KHz$, and $50 - 12.5KHz$ (see Cramton 1997).

We consider a private value model with two simultaneous auctions and different prizes. A bidder is informed about his valuation v and decides whether to participate in the higher prize auction or the lower prize auction. While in the low-prize auction the bidder's valuation is equal to his type v , in the high-prize auction his valuation is equal to av , where $a > 1$ (or for more general high-prize transformation where the valuation of the high-prize is $g(v) > v$). Börgers and Dustmann (2003, 2005) show that in UK frequency auctions, the bidders evaluate the high prize as equal to 1.5 of the low prize. In this case we can assume that $a = 1.5$. Similarly, Cramton (1997) shows that in the US, for FCC auctions for nationwide narrowband frequencies, the final winning bids for all licenses in terms of $\$/MHz\text{-pop}$ is almost identical although some licences have a different bandwidth.² Bidders evaluate a licence with $50KHz$ bandwidth as 50% of licence with $100KHz$ bandwidth (namely $a = 2$).³ If we limit the bidders to apply only to one auction,⁴ then the intuition about the bidder's behavior will be ambiguous. A high valuation bidder may believe he should participate in the lower prize auction since as a strong bidder, the

²The $\$/MHz\text{-pop}$ is a common measure used in frequency auctions that represent the ratio between total payment for the license and the multiplication of frequency bandwidth in terms of MHz and population size.

³According to Table 1 in Cramton (1997), the maximum difference between the winning bids in terms of $\$/MHz\text{-pop}$ was 8%.

⁴In many public goods auctions, the authorities limit the bidder's participation by quantity restriction or by blocking incumbent bidders, e.g. in US FCC auctions (Cramton 1997) and in UK third generation cellular auctions (Börgers and Dustmann 2003).

probability of winning is high. On the other hand, a low valuation bidder may take a chance and participate in the high prize auction hoping to be the sole participant and win the higher prize. We show that a separating equilibrium exists where a bidder of type higher than a cutoff c will participate in the higher prize auction while a bidder of type below the cutoff c will participate in the lower prize auction. Moreover, by properly setting the minimum bid (reservation price) in the auction with the higher prize, the seller can control the value of the cutoff type c . In addition, we show that a weak-separating equilibrium exists in the sense that for a certain cutoff \underline{c} , bidders with a type above this cutoff will attend the higher prize auction while bidders with a type below this cutoff will randomize between the two auctions, with a higher probability to join the low prize auction.

The above discussion raises the question of how to determine the efficiency of the selling mechanism? Although authorities may wish to support a new competitor and limit the incumbents, they may also want to consider the efficiency of the selling mechanism. In the auctions literature, the term efficiency is frequently used to describe the auction's outcome in terms of allocating the objects to the bidders with the highest valuation. The notion of efficiency may also include the seller. If the seller's reservation price is zero and he announces a minimum participation bid, then a no-selling outcome may occur if all the bidders' valuations are below the minimum bid. In this case, we have a non-efficient allocation. We show that the probability of full efficiency under separating equilibrium is at least 0.368 and no more than 0.5.⁵ In addition, if $a > 1.648$ then the probability for

⁵0.5 is the probability of an efficient allocation in the case of identical prizes (i.e., $a = 1$) where bidders

an efficient outcome in a strong separating equilibrium is higher than this probability in a weak separating equilibrium.

Although the model we study deals with auction selection in the sense that a bidder has to choose between auctions, we deviate from the literature since the objects in the auctions are not identical and the equilibrium found splits the bidders into strong and weak subgroups. Moreover, in our setting there is only one auctioneer who sells his two objects in two different auctions and thus, there is no competition between auctioneers. This setting differs from other studies that consider auction selection. For example, McAfee (1993), Peters (1997) and Peters and Severinov (1997) study a model of auction selection with a stock of sellers, each with an identical single object for sale and a stock of buyers. The sellers independently announce the auction rules and the buyers independently choose in which auction they are going to participate. In this type of model, there are two levels of competition. The first is the competition between sellers trying to attract the buyers and in the second, the bidders, after they choose an auction, compete over the prize. In the limit case (where the number of auctioneers increases to infinity), all sellers announce the same auction mechanism which is equivalent to a second-price auction with zero reserve price, and the buyers randomize between the sellers. Hernando-Veciana (2005) shows the same result for a finite number of auctioneers when they consider a similar model with a finite set of possible minimum bids.

Burguet and Sákovics (1999) study a model of two sellers with identical objects who compete in a one-shot game to attract a given set of bidders. In their model, the sellers'

randomize between auctions (Burguet and Sákovics 1999).

strategy is to choose the reserve price. They find the symmetric equilibrium when the sellers set the same positive reserve price and the bidders randomize between the auctions. This result indicates that the case of large market with many sellers is different from the case of small number of sellers. A small number of sellers lead to an inefficient equilibrium in the sense that the reserve price is positive and the object may not be sold. Monderer and Tennenholtz (2004), analyze the subgame perfect equilibrium in the auctions selection game where the buyers have a specific risk aversion attitude. The present paper is similar to Burguet and Sákovic (1999) except for the assumption of identical objects. The case of identical objects is the extreme case in our model where $a = 1$.

This paper is organized as follows. In Section 2 we introduce the model. In Section 3 we show the existence of a strong separating equilibrium and its characteristics. In Section 4 we introduce the weak separating equilibrium. In Section 5 we discuss the efficiency issues of a separating equilibrium. In Section 6 we generalize the results for any selling mechanism and general high-price transformations and make some concluding remarks.

2 The Basic Model

Consider two similar (but not identical) non divisible objects H and L offered for sale in two simultaneous and independent second-price auctions henceforth referred to as ‘auction H’ and ‘auction L’ (later on we will generalize to any classical selling mechanism). There are n potential buyers with a private valuation for the object $v_i \in [0, 1]$, $i = 1, 2, \dots, n$. v_i is private information of buyer i and it is drawn according to a continuous distribution

function $F(v)$, $v \in [0, 1]$. If a buyer wins the object in auction L his gain is v_i while if he wins the object in auction H his valuation is av_i , where $a > 1$. We assume that each buyer can participate only in one of the auctions. Once buyer i is informed about his privately-known valuation for the object, v_i , he decides in which of the two auctions he will participate and then submits a bid $b \geq 0$. We assume that when a buyer decides in which of the auctions he will participate, he is not informed about the number of buyers who have decided to participate in the same auction. However, this assumption does not restrict our analysis and the results hold also for mechanisms where the auctioneer announces the number of bidders in each auction after the bidders choose their auction. Under this setting, a bidder's strategy space is $s_i = (j, b)$, $j = H, L$, $b \geq 0$, for $i = 1, 2, \dots, n$. Denote by $r_j \geq 0$, $j = H, L$ the minimum price in each auction announced by the auctioneer (reservation price). A bidder participating in auction j wins the object if his bid is the highest in that auction and he pays the second highest bid (or the reservation price r_j if there is no second highest bid in auction j).

3 Strong Separating Equilibrium

We will start by proving the existence of a strong separating equilibrium when a strong (high type) bidder participates in auction H , and a weak (low type) bidder participates in auction L . Next, we will show the conditions for which a separating equilibrium can exist. We will show that these conditions are mild and that this type of equilibrium is more likely to exist than might be expected.

Proposition 1 *Let $c \in (0, 1)$ satisfy the condition*

$$\int_{r_L}^c (1 - F(c) + F(s))^{n-1} ds = F^{n-1}(c) (ac - r_H) \quad (1)$$

and the constraint

$$1 \leq aF^{n-1}(c). \quad (2)$$

Then, if $ar_L < r_H$, the strategy

$$s(v) = \begin{cases} j = L, & b(v) = v, & \text{if } r_L \leq v \leq c, \\ j = H, & b(v) = av, & \text{if } c < v \leq 1, \end{cases} \quad (3)$$

forms a symmetric equilibrium where a bidder of type below r_L does not participate in any auction.

Proof: See Appendix A. \square

The pivotal type c is the type of bidder who is indifferent between participating in auction L and winning with certainty, or participating in auction H and winning only if he is the sole bidder attending auction H . Equation (1) is the condition that constrains the pivotal bidder's utility to be identical in both auctions. Observe that the equilibrium bid function (3) is the same as in a regular second-price auction. From the proof for Proposition 1 we can find that condition (2) is required for an equilibrium to exist. To gain more insight into this condition, we can present it as $c \leq acF^{n-1}(c)$. This indicates that the expected income for a bidder with a cutoff value when he participates in auction L is less than his income when he participates in auction H . Moreover, according to Myerson (1981), the expected payoff for a bidder in an auction is $\int_r^v P(t)dt + constant$

where $P(t)$ is the probability that type t will win the object (the constant appears in case of a participation fee or a reservation price). In our model, the payoff for type v in auctions L and H are given by $\int_{r_L}^v P^L(t)dt$ and $a \int_c^v P^H(t)dt$, respectively, where the constants are omitted. It is simple to verify that the marginal expected payoff for the pivotal type is 1 in auction L and $aF^{n-1}(c)$ in auction H . Thus, condition (2) implies that the cutoff type marginal revenue is higher in auction H . From Figure 1 we can see that the expected payoff for a bidder as a function of his type is continuous at $v = c$ (but not smooth) since the pivotal bidder is indifferent between participating in auction L or H . Furthermore, the slope of the payoff is sharper when $v > c$ than when $v < c$, which ensures that high type bidders will not try to play as a weak type by participating in auction L . If (2) is binding then the expected utility is smooth at the pivotal type since the slope is equal from both sides.

Condition (2) is misleading since it gives the impression that by merely increasing the value of a , this condition can be satisfied. However, the value of the pivotal type c depends on a through (1); increasing a affects the value of c and thus, the condition may not hold. If we increase a , we increase the value for bidders that participate in auction H but, at the same time, we have to increase the minimum price such that a bidder will not defect from auction L to auction H . Note that a, c and r_H , which satisfy the conditions in the proposition, are not unique and there is a range of possible parameters that generate a separating equilibrium and thus, a separating equilibrium is more likely to exist than expected.

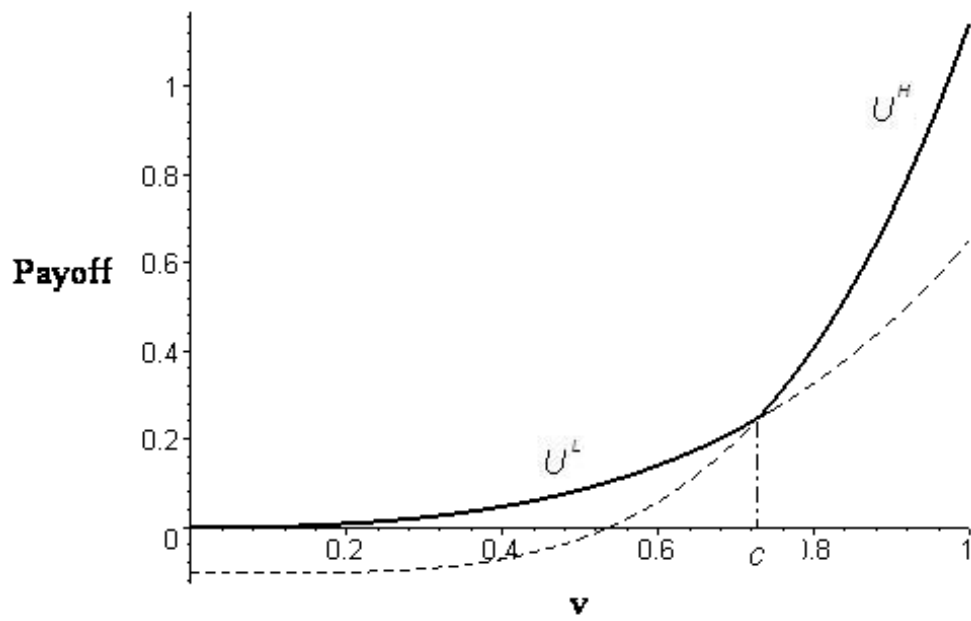


Figure 1: Bidder's expected utility as a function of his type v (bold curved line). The dashed lines represent the extension of the low and high utilities.

The condition $ar_L < r_H$ introduced in Proposition 1 guarantees that a weak bidder with a valuation below r_L (the minimum bid for participating in the auction L) will not try to participate in auction H . In other words, the reservation price in auction H is high enough or the reservation price in auction L is sufficiently low to completely block bidders with a value below r_L from participating in any auction.

While in Proposition 1 we show that given a cutoff that satisfies conditions (1) and (2) a separating equilibrium exists, there is still a question of whether or not a separating equilibrium exists for a given $F(v), a, n$. We can expect that the auctioneer has the freedom to set the minimum prices $r_A, A = L, H$, but that the value of the cutoff c is dictated by the bidders. We show that for any set of parameters a and n and distribution $F(v)$, choosing an appropriate minimum bid in both auctions can support a separating equilibrium. Moreover, if we let \underline{c} to be the solution to $aF^{n-1}(\underline{c}) = 1$, then for every $c \geq \underline{c}$ we can find minimum bids $r_A, A = L, H$ such that a separating equilibrium with pivotal type c exists.

Proposition 2 *For every a, n, F and $c \geq \underline{c}$, there exists r_L and r_H such that a separating equilibrium exists.*

Proof: Let $r_L < c \leq \underline{c}$ be any minimum bid and let $y(r_H) = \int_{r_L}^c (1 - F(c) + F(s))^{n-1} ds - F^{n-1}(c)(ac - r_H)$. Observe that $y(ar_L) < 0$ since $y(ar_L) = \int_{r_L}^c (1 - F(c) + F(s))^{n-1} ds - aF^{n-1}(c)(c - r_L) < c - r_L - aF^{n-1}(c)(c - r_L) = (1 - aF^{n-1}(c))(c - r_L) < 0$. In addition, $y'(r_H) = F^{n-1}(c) > 0$ and $y(ac) = \int_{r_L}^c (1 - F(c) + F(s))^{n-1} ds > 0$, which proves that there exists $r_H^0 = r_H^0(c)$ which solves (1) (i.e., $y(r_H^0) = 0$) and satisfies condition (2). Finally,

since $y(ar_L) < 0$, $y(r_H)$ is increasing and $y(r_H^0) = 0$, it follows that $ar_L < r_H^0$, which completes the proof. \square

Since we expect the auctioneer to set the minimum prices and the bidders to dictate the pivotal type through equilibrium, we should find out how the minimum prices would affect the cutoff.

Corollary 1 *The pivotal type c decreases with r_L and increases with r_H .*

Proof: Differentiating (1) with respect to r_L and r_H shows that $\partial c/\partial r_L < 0$ and $\partial c/\partial r_H > 0$. \square

It is surprising to find that by increasing r_L the cutoff decreases. The reason behind this finding is that since by increasing the minimum price in auction L we reduce the expected payment for the pivotal bidder and thus he is better off by participating in auction H . Thus, the pivotal type decreases.

Corollary 2 *The pivotal type c is decreasing with the value of auction H 's prize factor a .*

Proof: By fully differentiating (1) with respect to a , rearranging and using (2) we find that $\partial c/\partial a < 0$. \square

Corollary 2 shows that increasing a while holding the other parameters fixed will decrease c since more bidders closer to the pivotal type c will be better off by participating in auction H and thus, decrease c . While the monotonicity with respect to a is clear, the

behavior of the pivotal type with the number of bidders is ambiguous. We can give an example where c is not monotonic with the number of bidders n .

4 Weak Separating Equilibrium

In the previous section we demonstrated the existence of what we termed a *strong separating equilibrium* where bidders signal their type by choosing the auction they prefer. Now we show that there also exists an equilibrium where the separation is partial. In this case, bidders with a type above a given cutoff will participate in auction H with certainty while bidders with types below this cutoff will randomize between the two auctions. For simplicity, we assume $r_L = r_H = 0$, which excludes the possibility of a strong separating equilibrium.

Proposition 3 *Let $\hat{a} = a^{\frac{1}{n-1}}$ and \underline{c} be the solution of $aF^{n-1}(\underline{c}) = 1$. Then, there exists an equilibrium where bidders with value $v > \underline{c}$ participate in auction H and bidders with value $v \leq \underline{c}$ will randomize and participate in auction L with probability $\alpha = \frac{\hat{a}}{1+\hat{a}}$ and, in auction H with probability $1 - \alpha = \frac{1}{1+\hat{a}}$. Once a bidder participates in an auction he bids his value v in auction L and av in auction H .*

Proof: See Appendix B. \square

In Proposition 3 we found that the probability that a bidder of a type below \underline{c} will participate in auction L is $\alpha = \frac{\hat{a}}{1+\hat{a}} > \frac{1}{2}$, which indicates that a bidder with a low valuation is more likely to participate in auction L . In the extreme case, when $a = 1$, the cutoff

is $\underline{c} = 1$ and the probability of participating in each auction is 0.5. This boundary case appears when the objects are identical as we can find in Burguet and Sakovics (1999). We can observe that, according to the previous section, there exists also a strong separating equilibrium with the same cutoff \underline{c} . However, in this case r_H has to be positive.

5 Efficiency

Any mechanism selection with information asymmetry may lead to an inefficient outcome. The minimum bid is normally considered the reason for inefficiency in auction selection models (such as Burguet and Sákovics 1999) because it may lead to a situation where no bidder approaches any auction since all the bidders' types are below the minimum bid. However, there are two other types of inefficiencies that may occur in such models. One is when the randomization of bidders between auctions leads to a situation where although the bidder with the highest valuation wins an object, not necessarily does the bidder with the second highest valuation win the second object. In this case, a lower valuation wins, bringing about a different type of inefficiency. The second is when all the bidders apply to only one of the auctions so that the other object is not sold.

For simplicity, we assume that the lower reservation price is zero, i.e., $r_L = 0$. This assumption ensures that no inefficiency will arise by the minimum bid. Although we set $r_L = 0$, there are actually three types of possible inefficiencies in a separating equilibrium:

- 1) If more than one bidder has a type higher than cutoff c (but not all the n bidders) then the highest type will win the higher prize. However, the second highest type will

not win and the prize will go to a lower type that participates in auction L . 2) If there is no bidder attending auction L , the prize L will not be sold. 3) When no bidder attends auction H no one gets the higher prize. Observe that in Burguet and Sákovics (1999) and other auction selection models, there is no difference between the second and third type of inefficiencies since the objects are identical. In this section, we will find the probability of an efficient outcome and characterize the level of inefficiency for strong- and weak-separating equilibria. In addition, we will find the probability of not selling both objects.

5.1 Strong Separating Equilibrium

The event "efficient outcome" occurs when a single bidder has a value above c . One may expect that by increasing cutoff c we reduce the risk of having more than one bidder in that range. However, by increasing c we also increase the risk that no bidder will have a value above c and thus we increase the risk of the third type of inefficiency. In the next proposition we will characterize the optimal separating equilibrium that maximizes the probability of an efficient outcome.

Define $a^* = \left(\frac{n-1}{n}\right)^{-(n-1)}$ and let c^* be the optimal cutoff that maximizes the probability of an efficient outcome.

Proposition 4 *The optimal c^* is given by*

$$c^* = \begin{cases} F^{-1}\left(\frac{n-1}{n}\right), & \text{if } a \geq a^*, \\ \underline{c}, & \text{if } 1 < a < a^*, \end{cases}$$

where F^{-1} is the inverse function of F .

Proof: Given a cutoff c , the probability of an efficient outcome is $nF^{n-1}(c)(1 - F(c))$. To maximize this probability, the auctioneer can choose an optimal cutoff $c = F^{-1}(\frac{n-1}{n})$. However, the optimal cutoff should satisfy the equilibrium condition (2), which in this case is $a(\frac{n-1}{n})^{n-1} \geq 1$. Thus, if $a \geq a^*$ (2) is satisfied; otherwise \underline{c} is the maximizer where the constraint (2) is binding. \square

The optimal cutoff for the case where $a \geq a^*$ divides the range of types $[0, 1]$ such that the probability of an individual bidder to have a type above c is always $1/n$, independently of the distribution shape. Since $a^* = (\frac{n-1}{n})^{-(n-1)}$ is increasing with n and approaching e , if $a \geq e$ we are sure that the condition $a \geq a^*$ is satisfied for every distribution and number of bidders. If $a < 2$ the condition is never satisfied and thus, \underline{c} is optimal. Observe that the probability of an optimal efficient outcome when $a \geq e$ is $(\frac{n-1}{n})^{n-1}$ which starts from 0.5 for two bidders and decreases to $\frac{1}{e} \approx 0.368$. Thus, the probability for an efficient outcome is bounded between 0.368 to 0.5 and always decreases with the number of bidders n . In the case of identical objects the probability for an efficient outcome is 0.5.⁶

A question with respect to efficiency that now arises is: what is the probability that both of the prizes will be sold? Of course, this situation may not occur if all the bidders'

⁶In Burguet and Sákovics (1999), an efficient outcome occurs when the bidder with the highest type participates in one auction and the bidder with the second highest type participates in the other auction. Since a bidder's probability of participating in an auction is 0.5, the probability that the two highest types will participate in different auctions is 0.5.

valuations are above or below the cutoff c . The probability that both prizes will be sold is $1 - F^n(c) - (1 - F(c))^n$. The optimal cutoff that maximizes this probability is $F(c) = 0.5$ or $c = F^{-1}(0.5)$. However, condition (2) implies that $a \geq 2^{n-1}$ and obviously, if n is large enough, we will violate this condition. Thus, if $a < 2^{n-1}$ we will choose $c = \underline{c}$ to maximize the probability of selling both prizes. The maximum probability is given by $1 - a^{-\frac{n}{n-1}} - (1 - a^{-\frac{1}{n-1}})^n$ and converging to $1 - \frac{1}{a}$ as n increases. Surprisingly, the probability of selling both objects is not necessarily monotonic with n . Moreover, we may expect that by increasing the number of bidders the probability to sell both items will approach 1. However, this probability is strictly below 1 and converges to $1 - \frac{1}{a}$, since \underline{c} is increasing with n and less bidders are likely to have a type above the cutoff. Thus, if we want to maximize the probability of selling both objects, a separating equilibrium might be not worthwhile when a is low and we have many bidders.

5.2 Weak Separating Equilibrium

In the case of a weak separating equilibrium, an efficient outcome occurs when the highest type is above the cutoff \underline{c} or, he is below \underline{c} and participates in auction H while the second highest type is below \underline{c} and participates in auction L . The probability for an efficient outcome is $nF^{n-1}(\underline{c})(1 - F(\underline{c}))\alpha + F^n(\underline{c})\alpha(1 - \alpha)$ where α (defined in Proposition 3) is the probability that a bidder will participate in auction L . In the limit case when n is increasing to infinity, $\underline{c} = 1$ and thus, we may expect that efficiency is exactly $\frac{1}{4}$. However, the convergence rate is constrained by (2) such that $F^{n-1}(\underline{c}) = 1/a$. Therefore, while n

is increasing, there is always a positive probability to have bidders with valuation above the cutoff and thus, the limit is $\frac{\ln a}{2a} + \frac{1}{4a}$.⁷

Next we compare the probability of an efficient outcome between the strong and weak separating equilibria when we consider the same pivotal type \underline{c} in both equilibria.

Proposition 5 *If $a > \sqrt{e}$ then the probability for an efficient outcome in a strong separating equilibrium is higher than in a weak separating equilibrium.*

Proof: Observe that $nF^{n-1}(\underline{c})(1 - F(\underline{c}))$ is lower than the optimal probability for efficient outcome in strong separating equilibrium (which is at least $nF^{n-1}(\underline{c})(1 - F(\underline{c}))$ as we can learn from Proposition 4). Thus, the difference in probability for an efficient outcome between a strong and weak separating equilibrium is at least $nF^{n-1}(\underline{c})(1 - F(\underline{c})) - [nF^{n-1}(\underline{c})(1 - F(\underline{c}))\alpha + F^n(\underline{c})\alpha(1 - \alpha)] = F^{n-1}(\underline{c})(1 - \alpha)[n(1 - F(\underline{c})) - F(\underline{c})\alpha]$. To prove this result, it is sufficient to show that $n(1 - F(\underline{c})) - F(\underline{c})\alpha = n(1 - 1/a^{1/(n-1)}) - 1/(1 + a^{1/(n-1)}) > 0$. After multiplying the last inequality by the positive term $a^{1/(n-1)}(1 + a^{1/(n-1)})$ it remains to show that $n(a^{2/(n-1)} - 1) - a^{1/(n-1)} > 0$. Thus, the probability for an efficient outcome in a strong separating equilibrium is higher when

$$a > \left(\frac{1 + \sqrt{1 + 4n^2}}{2n} \right)^{n-1}. \quad (4)$$

Observe that the right-hand side of (4) monotonically increases to \sqrt{e} . \square

We can observe that $a > \sqrt{e} \approx 1.6487$ is an upper bound and thus, even if a is below this bound, when there are not too many bidders, a strong separating equilibrium is still more efficient. For example, in the case of less than six bidders, $a = 1.5$ suffices.

⁷Notice that for $a = 1$ the probability is indeed $1/4$.

We will find now the probability that the seller will sell both objects. The probability to sell both objects is $1 - [\alpha F(\underline{c})]^n - [1 - \alpha F(\underline{c})]^n = 1 - \frac{1 + a^{\frac{n}{n-1}}}{\left(1 + a^{\frac{1}{n-1}}\right)^n}$. This probability increases and rapidly converges to 1 (convergence rate is 2^{-n}). Thus, except for very small n , the probability of selling both objects is higher in a weak separating equilibrium.

6 Generalization of the Model and Concluding Comments

In this section we consider two possible generalizations for our model. First, we will question the robustness of the results when the valuation for the prize in auction H is $g(v)$ and not just linear. Second, we consider other selling mechanisms and not just the second-price auction.

6.1 General High-prize Transformation

Assume that the relation between a bidder's type v and his high-prize valuation is given by $g(v)$ where $g(v) > 1$ is monotonically increasing and concave. Assume that c satisfies the pivotal condition $\int_{r_L}^c (1 - F(c) + F(s))^{n-1} ds = F^{n-1}(c) (g(c) - r_H)$ and in addition that $g'(c)F^{n-1}(c) > 1$ and $g(r_L) < r_H$. Then, Proposition 1 holds. The arguments in this case are similar to the discussion in Section 3. It is simple to verify that $U^H(v|v) = F^{n-1}(c) (g(c) - r_H) + \int_c^v g'(s)F^{n-1}(s)ds$ and thus the marginal payoff for the pivotal bidder when he participates in auction H is $g'(c)F^{n-1}(c)$. The marginal payoff for

this bidder when participating in auction L is 1. Similarly to (2), the equilibrium condition $g'(c)F^{n-1}(c) > 1$, requires that for a high type, participating in auction H is better than participating in L . As in Proposition 1, the pivotal bidder is indifferent between the H and L auctions.

6.2 General Selling Mechanisms

Until now we considered only second-price auctions. However, the results of this paper are more general and are not limited to any specific type of auction. We may also consider different selling mechanisms for the different auctions. The generalization follows from the Revenue Equivalence Theorem and its generalization for a random number of bidders (see Myerson (1981) and Riley and Samuelson (1981)).

Let $b^L(v)$ be a bidder's equilibrium strategy in a given classic selling mechanism L with a minimum bid r_L , a distribution of types $F(v)/F(c)$, $v \in [0, c]$ and a random number of bidders distributed according to a binomial distribution with parameters n (number of trials) and $F(c)$ (probability of success). Similarly, let $b^H(v)$ be a bidder's equilibrium strategy in a given classic selling mechanism H with a minimum bid r_H , a distribution of types $(F(v) - F(c))/(1 - F(c))$ and a random number of bidders distributed according to a binomial distribution with parameters n and $1 - F(c)$. Let \hat{r}_L be the lowest type that will participate in auction L and $E(c, r_H)$ be the expected payment for a bidder of type c in auction H .⁸

⁸In first- and second-price auctions, $E(c, r_H) = F^{n-1}(c)r_H$.

Proposition 6 *Let $c \in (0, 1)$ satisfy the pivotal condition*

$$\int_{\hat{r}_L}^c (1 - F(c) + F(s))^{n-1} ds = acF^{n-1}(c) - E(c, r_H) \quad (5)$$

and the equilibrium constraint

$$1 \leq aF^{n-1}(c). \quad (6)$$

Then if $a\hat{r}_L < E(c, r_H)$ the strategies

$$s(v) = \begin{cases} j = L, & b(v) = b^L(v) & \text{if } \hat{r}_L \leq v \leq c, \\ j = H & b = b^H(v) & \text{if } c < v \leq 1, \end{cases} \quad (7)$$

form a symmetric equilibrium where a bidder of type below r_L does not participate in any auction..

Proof: See Appendix C. \square

6.3 Concluding Comments

We introduced a model where two nonidentical objects are being sold in a simultaneous independent auction where the bidders are allowed to participate in only one of those auctions. We found that there exists a separating equilibrium where bidders signal their type by participating in one of the auctions. Moreover, we found that there exists an equilibrium where bidders imperfectly signal their type by randomizing between auctions. The model we introduced has implications for authorities who wish to sell many public properties simultaneously and would like, in addition to efficiency, to limit monopolistic behavior by giving to weak players a chance to enter the market and thus, increase competition.

A Proof of Proposition 1

Proof of Proposition 1

Let $U^A(\hat{v}|v)$ be a bidder's expected utility given that his type is v , he acts as if his type is \hat{v} and he participates in auction $A = L, H$ and the other $n - 1$ bidders playing according to (3).⁹ Since the mechanism used in auction A is a second-price auction, a bidder's optimal strategy is to submit a bid equal to his type. Namely, in auction L the bidder submits $b = v$ and in auction H he submits $b = av$. Thus, $U^A(v|v)$ is the equilibrium expected revenue in auction A . It is left to show that no bidder will defect from one auction to the other one. In terms of incentive compatibility conditions, we require that for every v and \hat{v} ,

$$U^L(v|v \leq c) \geq U^H(\hat{v}|v \leq c), \quad (8)$$

$$U^L(\hat{v}|v \geq c) \leq U^H(v|v \geq c). \quad (9)$$

Obviously, if $v < r_H/a$, a bidder will not participate in auction H and if $v < r_L$ he will not participate in any auction. Let $P^L(v) = (1 - F(c) + F(v))^{n-1}$ be the equilibrium probability that a bidder of type v , $r_L \leq v \leq c$ will win in auction L (if $v < r_L$ then $P^L(v) = 0$). By a standard argument

$$U^L(v|v \leq c) = \int_0^v P^L(s)ds = \int_{r_L}^v (1 - F(c) + F(s))^{n-1} ds.$$

Similarly, since $P^H(v) = F^{n-1}(v)$ is the probability that a bidder of type $v \geq c$ will win

⁹We use the notation $U^L(\hat{v}|v \leq c)$ for a bidder's revenue when his type is v , $v \leq c$.

in auction H ,

$$U^H(v|v \geq c) = \text{const} + a \int_c^v P^H(s)ds = (ac - r_H)F^{n-1}(c) + a \int_c^v F^{n-1}(s)ds.$$

The constant $(ac - r_H)F^{n-1}(c)$ appears since a bidder's revenue in auction H is not zero even if his type is c . The probability $P^H(v)$ is multiplied by a since the gain is multiplied by a (i.e., av). The constant $(ac - r_H)F^{n-1}(c)$ is the expected utility if the bidder's type is $v = c$. Since auction L is a second-price auction, and

$$\begin{aligned} U^L(\hat{v} \leq c|v \geq c) &= U^L(\hat{v} \leq c|\hat{v} \leq c) + (v - \hat{v}) \Pr(\text{wins with bid } v) = \\ &= \int_{r_L}^{\hat{v}} (1 - F(c) + F(s))^{n-1} ds + (v - \hat{v})(1 - F(c) + F(\hat{v}))^{n-1} \leq \\ &\leq \int_{r_L}^c (1 - F(c) + F(s))^{n-1} ds + (v - c)(1 - F(c) + F(\hat{v}))^{n-1}, \end{aligned}$$

it follows that

$$\begin{aligned} U^L(\hat{v} \leq c|v \geq c) &\leq \int_{r_L}^c (1 - F(c) + F(s))^{n-1} ds + v - c = \\ &= (ac - r_H)F^{n-1}(c) + v - c \leq (ac - r_H)F^{n-1}(c) + (v - c)aF^{n-1}(c) \leq \\ &\leq (ac - r_H)F^{n-1}(c) + a \int_c^v F^{n-1}(s)ds = U^H(v|v \geq c) \end{aligned}$$

and thus (9) is satisfied. The second equality and the subsequent inequality is derived from (1) and (2). Since H is a second-price auction, it follows that

$$U^H(\hat{v} > c|v \leq c) \leq U^H(\hat{v} = c|v \leq c) = (av - r_H)F^{n-1}(c).$$

To complete the proof, we show that $U^L(v|v \leq c) = \int_{r_L}^v (1 - F(c) + F(s))^{n-1} ds \geq (av - r_H)F^{n-1}(c)$. Define $h(v) = \int_{r_L}^v (1 - F(c) + F(s))^{n-1} ds - (av - r_H)F^{n-1}(c)$ and observe that

by the assumption $ar_L < r_H$ we have $h(r_L) = -(ar_L - r_H)F^{n-1}(c) > 0$. By the definition of c in equation (1) we have $h(c) = 0$. Thus, if $h(v)$ is non-increasing, the condition $h(v) \geq 0$ is satisfied. Since $h'(v) = (1 - F(c) + F(v))^{n-1} - aF^{n-1}(c) \leq 1 - aF^{n-1}(c)$ it follows by (2) that $h'(v) \leq 0$. Thus,

$$U^L(v|v \leq c) = \int_{r_L}^v (1 - F(c) + F(s))^{n-1} ds \geq (av - r_H)F^{n-1}(c) \geq U^H(\hat{v} > c|v \leq c)$$

and (8) is satisfied. \square

B Proof of Proposition 3

Assume that bidders with value $v \leq \underline{c}$ participate in auction L with probability α and in auction H with probability $1 - \alpha$. Since the auction is a second-price auction, once a bidder with value $v \leq \underline{c}$ approaches one of the auctions, he will bid his value v in auction L and av in auction H . A bidder's expected payoff is given by $\int_0^v \Pr^L(t)dt$ and $a \int_0^v \Pr^H(t)dt$ where $\Pr^i(t)$ $i = L, H$ is the probability that the bidder will win in auction i (see, for example, Myerson (1981)). In a mixed-strategy equilibrium, a bidder of type $v \leq c$ should be indifferent between the two auctions. Thus $\int_0^v \Pr^L(t)dt = a \int_0^v \Pr^H(t)dt$ for every $v \leq \underline{c}$. Since for $v = 0$ both expected payoffs equal zero, it remains to show that the derivative is equal for every $v \leq \underline{c}$, i.e., $\Pr^L(v) = a \Pr^H(v)$. The probability of a bidder with value $v \leq \underline{c}$ to win is $\Pr^L(v) = [1 - \alpha F(\underline{c}) + \alpha F(v)]^{n-1}$ in auction L and $\Pr^H(v) = [\alpha F(\underline{c}) + (1 - \alpha)F(v)]^{n-1}$ in auction H . After rearranging we have $\Pr^L(v) = \left[\frac{\hat{a}}{1+\hat{a}}(1 + F(v)) \right]^{n-1} = a \Pr^H(v)$. To complete the proof it remains to show that a bidder

with value $v > \underline{c}$ will not deviate and participate in auction L . Observe that $\Pr^H(v) = F(v)^{n-1}$ if $v > \underline{c}$ and $\Pr^H(v) = \left[\frac{\hat{a}}{1+\hat{a}}(1+F(v))\right]^{n-1}$ if $v \leq \underline{c}$. Thus $U^H(v|v \geq \underline{c}) = a \int_0^v \Pr^H(t)dt = \left[\frac{\hat{a}}{1+\hat{a}}\right]^{n-1} \int_0^{\underline{c}} (1+F(s))^{n-1} ds + a \int_{\underline{c}}^v F(s)^{n-1} ds$ and $U^L(v|v \geq \underline{c}) = U^L(\underline{c}|v = \underline{c}) + (v - \underline{c}) = \int_0^{\underline{c}} \Pr^L(t)dt + (v - \underline{c}) = \left[\frac{\hat{a}}{1+\hat{a}}\right]^{n-1} \int_0^{\underline{c}} (1+F(s))^{n-1} ds + (v - \underline{c})$. To prove that $U^H(v|v \geq \underline{c}) \geq U^L(v|v \geq \underline{c})$ it remains to show that $a \int_{\underline{c}}^v F(s)^{n-1} ds \geq (v - \underline{c})$. Since $a \int_{\underline{c}}^v F(s)^{n-1} ds \geq a(v - \underline{c})F(\underline{c})^{n-1} = (v - \underline{c})$ the proof is completed. \square

C Proof of Proposition 6

Calculating the expected payoff for a bidder in auction L , given that his type is v and there are a random number of competing bidders distributed according to binomial distribution with $n - 1$ trials, the probability of success $F(c)$ and distribution $F(v)/F(c)$, $v \in [0, c]$ will give exactly the same results as in the case of a second price auction shown in the previous section. This follows from the revenue equivalence theorem that is valid also for a random number of bidders (see Milgrom 2004).¹⁰ Similar arguments hold for the expected payoff in auction H , however, here we need to be a bit more careful since revenue equivalence assumes that the bidder with the lowest valuation will obtain zero expected payoff. In auction H , the expected payoff for the bidder with the lowest valuation c (or actually ac) is equal to the expected payoff of the highest valuation in auction L and is strictly positive. The proof continues similarly to the proof of Proposition 1. \square

¹⁰A different statement of the RET said that the expect payoff for bidder with type v is independent of the auction mechanism.

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