

Asymptotic Analysis of Large Auctions

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Abstract

We study private-value auctions with n bidders, where n is a large number. We use asymptotic techniques to calculate explicit approximations of the equilibrium bids and of the seller's revenue in symmetric k -price auction ($k = 1, 2, \dots$) with risk-averse bidders, and use these explicit approximations to show that the effect of risk aversion on the seller's revenue is $O(1/n^2)$ small. We also prove that second-price auctions with asymmetric bidders are $O(1/n^2)$ revenue equivalent to a large class of standard auctions.

Keywords: Large auctions, asymmetric auctions, risk-averse bidders, asymptotic methods, revenue equivalence.

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1 Introduction

Many auctions, particularly those which have recently begun to appear on the internet, have a large number of bidders. The standard approach to study large auctions has been to consider their limit as n , the number of bidders, approaches infinity.¹ Using this approach, it has been shown for quite general conditions that as n goes to infinity, the bid approaches the true value, the seller's expected revenue approaches the maximal possible value, and the auction becomes efficient.² Most of the studies that adopted this approach, however, do not provide the rate of convergence to the limit, i.e., a bound on the difference between the limiting value and the value at a finite n . Therefore, it is not clear how large n should be (5, 10, 100?) in order for the auction "to be large" (i.e., in order for the limiting results obtained for $n = \infty$ to be applicable). Results of this type were obtained by Satterthwaite and Williams (1989), who showed that the rate of convergence of the bid to the true value in a double auction is $O(1/m)$, where m is the number of traders on each side of the market, and by Rustichini, Satterthwaite and Williams (1994), who showed that the rate of convergence of the bid to the true value in a k -double auction is $O(1/m)$ and the corresponding inefficiency is $O(1/m^2)$.³

In this study we use asymptotic analysis techniques in order to go beyond rate-of-convergence results, namely, we utilize the existence of the large parameter n to calculate

¹This is the approach, for example, in Wilson (1976), Pesendorfer and Swinkels (1997), Kremer (1999), Swinkels (1999).

²See, for example, Swinkels (2001) and Bali and Jackson (2001).

³Hong and Shum (2004) calculated the convergence rate in common-value multi-unit first-price auctions.

explicitly the leading-order deviation from the limiting value of the equilibrium bids and of the seller's revenue, regardless of whether bidders are symmetric or asymmetric, risk-neutral or risk-averse. Since the leading-order deviation terms are $O(1/n)$, our asymptotic results are $O(1/n^2)$ accurate. Hence, our asymptotic results become valid at much smaller values of n than the limiting results, which are only $O(1/n)$ accurate.

In our previous studies we used perturbation analysis techniques to analyze auctions with weakly asymmetric bidders (Fibich and Gaviols (2003), Fibich, Gaviols and Sela, (2004)) and with weakly risk-averse bidders (Fibich, Gaviols and Sela, (2002)). The present paper improves on those studies in two aspects:

1. From the economic theory aspect, in those studies we had to assume that the level of asymmetry (or risk-aversion) is small, in order to be able to expand the solution in the small asymmetry (or risk-aversion) parameter. The results of this study are stronger, since we do not need to make such assumptions, as we can utilize the existence of the large parameter n .
2. From the mathematical analysis aspect, in our previous studies we used perturbation techniques that “essentially” amounts to Taylor expansions in a small parameter that lead to convergent sums. In contrast, in this study we use asymptotic methods (e.g., Laplace method for evaluation of integrals) which typically lead to divergent sums if carried out to all orders (see, e.g., Murray, 1984). To the best of our knowledge, these asymptotic techniques have not been used in auction theory so far. These techniques, as well as other asymptotic methods (WKB, method of steepest descent, etc.) are

quite likely to be useful in other economic problems where a large parameter exist, e.g., multi-unit auctions with many units (Jackson and Kremer, 2004; Jackson and Kremer, 2005).

Since the pioneering work of Vickrey (1961) who established the revenue equivalence of the classical private-value auctions (first-price, second-price, English, Dutch), a considerable research effort has been devoted to revenue ranking of different auction mechanisms. Vickrey's result was generalized twenty years later by the Revenue Equivalence Theorem (Riley and Samuelson (1981) and Myerson (1981)) according to which the seller's revenue is the same for a wide class of private-value auctions with symmetric and risk-neutral bidders. Private value auctions are, however, in general not revenue equivalent when bidders are asymmetric (Marshall et al. (1994), Maskin and Riley (2000)) or risk-averse (Maskin and Riley (1984), Matthews (1987)). As we mentioned, previous studies showed that under quite general conditions, such auctions become revenue equivalent as n approaches infinity. In this work we prove a stronger result, namely, that independently of whether bidders are symmetric or asymmetric, risk-neutral or risk-averse, for large classes of standard auctions the $O(1/n)$ deviation of the revenue from the limiting revenue is also independent of the auction mechanism. In other words, the revenue difference among large auctions with n bidders is at most $O(1/n^2)$. This result suggests that revenue ranking of large auctions is probably more of academic interest than of practical value.⁴

⁴In fact, in all the numerical examples that we have tested (see, e.g., Table 1) we found that already for auctions with six bidders, the revenue difference between first- and second-price auctions is in the fourth or fifth digit.

The paper is organized as follows. In Section 2 we calculate asymptotic approximations of the equilibrium bids and of the seller’s revenue in large symmetric k -price auctions ($k = 1, 2, \dots$) with risk-averse bidders, and show that the differences in the equilibrium bids and in the seller’s revenue between risk-neutral and risk-averse bidders are only $O(1/n^2)$. Therefore, we conclude that all large k -price auctions with risk-averse or risk-neutral bidders are $O(1/n^2)$ revenue equivalent. In Section 3 we calculate asymptotic approximations of the revenue in large asymmetric auctions. As was pointed out by Swinkels (2001), while in large asymmetric auctions “players’ values may come from very different distributions, their environments, and thus their optimal behavior with any given valuation, may be very similar”. Since the environment (competition) that players i and j face differs by one out of $n - 1$ players (i is facing j but not i and vice versa), one can expect the resulting revenue differences among different auction mechanisms to be $O(1/n)$. However, our results show that a large class of auctions with asymmetric bidders are $O(1/n^2)$ revenue equivalent. Finally, numerical examples suggest that the asymptotic approximations derived in this study are quite accurate for auctions with as little as $n = 6$ players. Therefore, the number of bidders n does not have to be really large for our asymptotic results to be valid. Section 4 concludes, and the Appendix contains proofs omitted from the main body of the paper.

2 Large symmetric auctions

Consider a large number ($n \gg 1$) of bidders who are competing for a single object. The bidders are symmetric such that the valuation v_i of bidder i for the object is independently distributed according to a common distribution function $F(v)$ on the interval $[0, 1]$. We denote by $f = F'$ the corresponding density function. We assume that F is twice continuously differentiable and that $f > 0$ in $[0, 1]$. We assume that each bidder's utility is given by a function $U(v - b)$, which is twice continuously differentiable, normalized to have zero utility at zero, and monotonically increasing, i.e.,

$$U(x) \in \mathcal{C}^2, \quad U(0) = 0, \quad U'(x) > 0. \quad (1)$$

Since we place no restriction on U'' , the results of this section hold for risk-averse, risk-neutral, and risk-loving bidders.

2.1 First-price auctions

In the case of first-price auctions, the inverse equilibrium bids satisfy

$$v'(b) = \frac{1}{n-1} \frac{F(v(b)) U'(v(b) - b)}{f(v(b)) U(v(b) - b)}, \quad v(0) = 0. \quad (2)$$

In this setup, there are no explicit formulae for the equilibrium bids and for the revenue, except for some special cases. Recently, Fibich, Gaviols and Sela (2002) obtained explicit approximations of the equilibrium bids for the case of weak risk aversion, by using perturbation methods to expand the solution in the small risk-aversion parameter. In contrast, here we use different mathematical techniques and expand the solution in the

large parameter n , without making the assumption that risk aversion is weak.⁵

Proposition 1 *Consider a symmetric first-price auction with n bidders with utility function $U(x)$ that satisfied Assumptions (1). Then, the equilibrium bid for a sufficiently large n is given by*

$$b(v) = v - \frac{1}{n-1} \frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right), \quad (3)$$

and the seller's expected revenue is given by

$$R[F] = 1 - \frac{2}{n} \frac{1}{f(1)} + O\left(\frac{1}{n^2}\right). \quad (4)$$

Proof. See Appendix B.

Proposition 1 shows that the differences in the equilibrium bids and in the seller's revenue between auctions with risk-neutral or with risk-averse bidders are at most $O(1/n^2)$. In other words, risk aversion only has (at most) an $O(1/n^2)$ effect on the equilibrium bids and on the revenues in symmetric first-price auctions.⁶

2.1.1 Going to the next-order

The results of Proposition 1 raise several questions:

1. Is the effect of risk-aversion truly $O(1/n^2)$, or is it even smaller?

⁵Caserta and de Vries (2002) used extreme value theory to derive an asymptotic expression for the revenue which is equivalent to (4). The result of Caserta and de Vries (2002) holds, however, only in the risk-neutral case, where an explicit expression for the revenue is available.

⁶Our conclusion that risk aversion has a negligible effect in large auctions is consistent with Dyer, Kagel and Levin (1989), who found in experiments with six bidders that the actual bids were very close to the theoretical risk-neutral equilibrium bid.

2. Can we estimate the constants in the $O(1/n^2)$ error terms?
3. To what extent do the results depend on Assumptions (1) for the utility function?

We can answer the first two questions by calculating explicitly the $O(1/n^2)$ terms:

Proposition 2 *Consider a symmetric first-price auction with n bidders with utility function $U(x)$ that satisfied Assumptions (1). Then, the equilibrium bid for a sufficiently large n is given by*

$$b(v) = v - \frac{1}{n-1} \frac{F(v)}{f(v)} + \frac{1}{(n-1)^2} \left[\frac{F(v)}{f(v)} - \frac{F^2(v)f'(v)}{f^3(v)} - \frac{F^2(v)}{2f^2(v)} \frac{U''(0)}{U'(0)} \right] + O\left(\frac{1}{n^3}\right), \quad (5)$$

and the seller's expected revenue is given by

$$R[F] = 1 - \frac{1}{n} \frac{2}{f(1)} + \frac{1}{n^2} \left[\frac{2}{f(1)} - \frac{3f'(1)}{f^3(1)} - \frac{1}{2f^2(1)} \frac{U''(0)}{U'(0)} \right] + O\left(\frac{1}{n^3}\right). \quad (6)$$

Proof. See Appendix C.

We thus see that risk-aversion has an $O(1/n^2)$ effect on the bid when $U''(0) \neq 0$, but a smaller effect if $U''(0) = 0$. As expected, the bids and revenue increase (decrease) for risk-averse (risk-loving) bidders. Note that the magnitude of risk-aversion effect is determined by the value of the absolute risk-aversion $-U''/U'$ at zero.

To answer the third question, we note that Assumptions (1) imply that $U'(0) < \infty$, and are thus invalid in the case of a CRRA utility function $U(x) = x^\alpha$, $0 < \alpha < 1$. In that case, risk-aversion turns out to have an $O(1/n)$ effect on the bid and revenue:

Lemma 1 *Consider a large symmetric first-price auction with n bidders with a CRRA utility function $U(x) = x^\alpha$ where $0 < \alpha < 1$. Then,*

$$b(v) = v - \frac{\alpha F(v)}{n f(v)} + O\left(\frac{1}{n^2}\right), \quad (7)$$

$$R[F] = 1 - \frac{1 + \alpha}{n f(1)} + O\left(\frac{1}{n^2}\right). \quad (8)$$

Proof. See Appendix D.

The observation that risk-aversion has a small effect on the revenue in large auctions is not surprising. Indeed, since in large auctions the bids are close to the values, one can approximate $U(v-b) \approx (v-b)U'(0)$, which is the risk-neutral case. However, this intuitive argument cannot tell us that the effect of risk-aversion decays as $1/n^2$ when $U'(0) > 0$, but as $1/n$ for CRRA bidders, etc.

2.2 k -price auctions

The results of Proposition 1 can be generalized to any k -price auction:⁷

Proposition 3 *Consider a symmetric k -price auction ($k = 1, 2, 3, \dots$), with n bidders with utility function $U(x)$ that satisfies Assumptions (1). Then, the equilibrium bid for a sufficiently large n is*

$$b(v) = v + \frac{k-2}{n-k} \frac{F(v)}{f(v)} + O\left(\frac{1}{n^2}\right), \quad (9)$$

and the seller's expected revenue is given by (4).

⁷In a k -price auction the bidder with the highest bid wins the auction and pays the k -th highest bid.

For more details on k -price auctions with risk averse bidders, see Monderer and Tennenholtz (2000).

Proof. See Appendix E.

Recall that in the risk-neutral case $U(x) = x$, the equilibrium bids in k -price auctions are given by (Wolfstetter, 1995)

$$b(v) = v + \frac{k-2}{n-k+1} \frac{F(v)}{f(v)}.$$

Comparison with equation (9) shows that in large symmetric k -price auctions, risk aversion only has an $O(1/n^2)$ effect on the equilibrium bids. Proposition 3 also shows that risk aversion only has an $O(1/n^2)$ effect on the revenue in large symmetric k -price auctions. Since all k -price auctions are revenue equivalent in the risk-neutral case, this implies, in particular, that all large symmetric k -price auctions, with risk-loving or risk-averse bidders, are $O(1/n^2)$ revenue equivalent.

Remark. As in the case of first-price auctions (see Section 2.1.1), we can calculate explicitly the $O(1/n^2)$ terms in order to see that the effect of risk-aversion is truly $O(1/n^2)$ when $U'(0) > 0$ and $U''(0) \neq 0$, that the effect of risk-aversion is only $O(1/n)$ for CRRA bidders, etc.

3 Asymmetric large auctions

We now consider the case of large asymmetric auctions with n risk-neutral bidders. The valuation of bidder i for the object v_i is independently distributed according to a distribution function $F_i(v)$ on the common interval $[0, 1]$. We denote by $f_i = F_i'$ the corresponding density function.

We assume that f_j , f'_j and f''_j are uniformly bounded, i.e., that there exist two positive constants $0 < m < M$, such that for every $v \in [0, 1]$ and for $j = 1, \dots, n$,

$$m \leq f_j(v) \leq M, \quad |f'_j(v)|, |f''_j(v)| \leq M. \quad (10)$$

In fact, for our results to hold, it is enough to require that (10) holds in a small neighborhood of $v = 1$.

We denote by

$$F_G(v) = \left(\prod_{i=1}^n F_i(v) \right)^{1/n}$$

the geometric average of the distribution functions and also denote $f_G(v) = F'_G(v)$.

3.1 Second-price auctions

Consider a large number ($n \gg 1$) of asymmetric bidders with distribution functions F_1, \dots, F_n who are competing for a single object in a second-price auction. In this case, the equilibrium bid functions are $b_i(v) = v$, and the seller's expected revenue is given by (see, e.g., Fibich and Gavious (2003))

$$R^{2\text{nd}}[F_1, \dots, F_n] = 1 - \int_0^1 \prod_{i=1}^n F_i(v) dv - \sum_{i=1}^n \int_0^1 (1 - F_i(v)) \prod_{\substack{j=1 \\ j \neq i}}^n F_j(v) dv. \quad (11)$$

Asymptotic expansion of $R^{2\text{nd}}$ in $1/n$ leads to the following result:

Proposition 4 *In an asymmetric large second-price auction with n bidders, the seller's expected revenue for a sufficiently large n is*

$$R^{2\text{nd}}[F_1, \dots, F_n] = 1 - \frac{2}{n} \frac{1}{f_G(1)} + O\left(\frac{1}{n^2}\right), \quad (12)$$

where $f_G(1) = F'_G(1) = \frac{1}{n} \sum_{j=1}^n f_j(1)$.

Proof. See Appendix F.

Cantillon (2003) showed that the revenue in asymmetric second-price auctions is always smaller than in a symmetric second-price auctions with the same number of players whose (symmetric) distribution function is F_G . Since $f_G(1)$ is the same in both cases, Proposition 4 shows that the revenue difference between the two cases is $O(1/n^2)$ small.

Example 1 *We consider a large asymmetric second-price auction where bidders are equally split to $n/2$ bidders with a distribution function $F_1(v) = v^{1/2}$ and $n/2$ bidders with a distribution function $F_2(v) = v^2$.*

In this case, $\frac{1}{f_G(1)} = 4/5$ and the asymptotic approximation (12) yields

$$R^{2nd} = 1 - \frac{8}{5n} + O\left(\frac{1}{n^2}\right). \quad (13)$$

By (11), the exact value of the seller's expected revenue is given by

$$R^{2nd} = 1 + \frac{n-1}{1.25n+1} - \frac{n/2}{1.25n-1} - \frac{n/2}{1.25n+0.5}.$$

Expanding this in $1/n$ gives

$$R^{2nd} = 1 - \frac{8}{5n} + \frac{104}{125n^2} + \dots,$$

which is in agreement with (13).

3.2 Asymptotic revenue equivalence

In the following we show that the asymptotic expression for the revenue in asymmetric second-price auctions (12) can be generalized to a large class of asymmetric auctions.⁸

Theorem 1 *Consider any auction mechanism that satisfies the following conditions:*

1. *All n players are risk neutral.*
2. *Player i 's valuation is private information to i and is drawn independently by a twice continuously differentiable distribution function $F_i(v)$ from a support $[0, 1]$.*
3. *The object is allocated to the player with the highest bid.*⁹
4. *Any player with valuation 0 expects zero surplus.*
5. *The maximal bid is identical for all bidders, i.e., $b_i(1) = b_j(1)$ for every i and j .*¹⁰
6. *The asymmetry among the equilibrium bids and their first, second, and third derivatives at the maximal value is at most $O(1/n)$, i.e., $b_i^{(k)}(1)/b_j^{(k)}(1) = 1 + O(1/n)$ for every i and j , and for $k = 1, 2, 3$.*¹¹

⁸Fibich, Gaviols and Sela (2003) showed that weakly-asymmetric auctions are asymptotically revenue equivalent, by expanding the revenue in the small asymmetry parameter. The result here is stronger, since we do not need to assume that the asymmetry among players is weak.

⁹In the symmetric case, assumption 3 is equivalent to the assumption that in equilibrium the object is allocated to the player with the highest valuation. This equivalence, however, does not hold in the asymmetric case since asymmetric auctions are not necessarily efficient.

¹⁰This condition is satisfied e.g., for asymmetric first- and second price auctions.

¹¹The motivation for this condition is as follows. Since the environment (competition) that players i

Then, the seller's expected revenue for a sufficiently large n is given by

$$R[F_1, \dots, F_n] = 1 - \frac{2}{n} \frac{1}{f_G(1)} + O\left(\frac{1}{n^2}\right). \quad (14)$$

Proof. See Appendix G.

As we have said, previous studies established the revenue equivalence of a large class of auctions in the limit as $n \rightarrow \infty$. The novelty in Theorem 1 is, thus, in showing that the $O(1/n)$ deviation from the limiting value is also independent of the auction mechanism. Clearly, in the case of symmetric auctions, the asymptotic relation (14) reduces to (4).

It is interesting to note that Condition 6 is not satisfied by asymmetric first-price auctions, since in that case $b'_i(1)/b'_j(1) = f_i(1)/f_j(1)$. Nevertheless, the result of Theorem 1 does hold for asymmetric first-price auctions, as is corroborated by our subsequent numerical examples. The reason for this is that one can split the integration segment $[0, 1]$ into two segments $[0, 1 - \epsilon]$ and $[1 - \epsilon, 1]$, and choose $\epsilon = \epsilon(n) \ll 1$, such that Condition 6 holds for $v = 1 - \epsilon$, and such that the contribution of the interval $[1 - \epsilon, 1]$ to the revenue is $O(1/n^2)$. Therefore, application of Theorem 1 to the interval $[0, 1 - \epsilon]$ will yield the result.

In order to examine the level of revenue equivalence when the number of players is not really large, in Table 1 we compare the expected revenue in first-price auctions, the expected revenue in second-price auction, and the asymptotic approximation for the revenue, for the case of 6 asymmetric bidders with various distributions. The first thing to

and j face differs by one out of $n - 1$ players (i is facing j but not i and vice versa), one could expect the resulting asymmetry among the bids to be $O(1/n)$.

note is that our asymptotic prediction

$$R[F_1, \dots, F_n] \approx 1 - \frac{2}{n} \frac{1}{f_G(1)} = 1 - \frac{2}{\sum_{i=1}^n f_i(1)}$$

is considerably more accurate¹² than the ‘old’ prediction

$$R[F_1, \dots, F_n] \approx \lim_{n \rightarrow \infty} R[F_1, \dots, F_n] = 1.$$

In addition, the good agreement between the asymptotic prediction and the actual revenues suggests (yet again) that the results of the asymptotic analysis are already valid for $n \approx 6$. Finally, we note that in all cases the differences between R^{1st} and R^{2nd} are only in the fourth or fifth digit. This observation suggests that, for all practical purposes, first-price and second-price auction with $n \geq 6$ bidders are revenue equivalent.

distributions	R^{1st}	R^{2nd}	$1 - \frac{2}{n} \frac{1}{f_G(1)}$	$R^{1st} - R^{2nd}$	$R^{1st} - \left(1 - \frac{2}{n} \frac{1}{f_G(1)}\right)$
$F_1 = F_2 = F_3 = F_4 = v^4, F_5 = F_6 = v^{1/2}$	0.877782	0.877778	0.882353	0.000004	-0.004571
$F_1 = F_2 = F_3 = v^4, F_4 = F_5 = F_6 = v^{1/2}$	0.84487	0.84483	0.851852	0.00004	-0.006982
$F_1 = F_2 = F_3 = v^2, F_4 = F_5 = F_6 = v^{1/2}$	0.751773	0.751697	0.733333	0.00008	0.01844
$F_i = v^i, i = 1, \dots, 6$	0.900224	0.900143	0.904762	0.000081	-0.004538

Table 1: Expected revenue $R^{1st}[F_1, \dots, F_6]$ and $R^{2nd}[F_1, \dots, F_6]$.

4 Concluding remarks

Fibich, Gaviols and Sela (2003) showed that for weakly-asymmetric auction, if ϵ is the small asymmetry parameter, all standard auctions are $O(\epsilon^2)$ revenue equivalent. Hence,

¹²by a factor of 13–27 for these examples

the revenue difference among weakly-asymmetric auctions is negligible. In this study we do not assume that the asymmetry is weak, yet we find that the revenue difference among large asymmetric auctions is also negligible. Together, these two studies suggest that in most cases asymmetry among bidders valuations plays a minor role in revenue ranking of auctions. Since asymmetry leads to a considerable increase in the complexity of the mathematical model, but has negligible effect on revenue ranking, it can probably be neglected in revenue ranking studies. A similar conclusion, however, cannot be applied to risk-aversion. Indeed, Fibich, Gavious and Sela (2002) showed that for symmetric auction with weakly-risk averse bidders, if ϵ is a small risk-aversion parameter, risk aversion has an $O(\epsilon)$ effect on the equilibrium bids, and then the revenue difference among auctions is $O(\epsilon)$. Therefore, risk-aversion can play an important role in revenue ranking of small auctions, but not in large ones.

If we try to generalize the results of this research, we can identify several unifying themes: 1) Auctions with a large number of bidders are considerably simpler to analyze than auctions with a small number of bidders, since the effects of various “complications” such as asymmetry, and risk-aversion are negligible. 2) The revenue ranking of large auctions is more of academic than of practical value. 3) Auctions with as few as six bidders can be considered as large auctions.

The last point raises the important question of how large the number of bidders (n) should be for our asymptotic results to be valid. Since our expansions have an $O(1/n^2)$ accuracy, roughly speaking, they have a 1% accuracy for 10 bidders. As the numerical examples in this study show, however, our asymptotic results are already quite accurate

even for values as low as $n = 6$. The issue of how small can n be for our asymptotic results to be valid can be resolved theoretically, in principle, by calculating explicitly the next $O(1/n^2)$ term in the asymptotic expansion, but this is beyond the scope of the present study.

A Auxiliary Lemmas

Lemma 2 *Let $n \gg 1$, let $b(v) = v + (1/n)B_1(v) + (1/n^2)B_2(v) + O(1/n^3)$, and let $v(b) = b + \frac{1}{n}v_1(b) + \frac{1}{n^2}v_2(b) + O(1/n^3)$ be the inverse function of $b(v)$. Then,*

$$B_1(v) = -v_1(v), \quad B_2(v) = -B_1(v)v_1'(v) - v_2(v). \quad (15)$$

Proof. We substitute the two expansions into the identity $v \equiv v(b(v))$ and expand in $1/n$:

$$\begin{aligned} v &= v(b(v)) = b(v) + \frac{1}{n}v_1(b(v)) + \frac{1}{n^2}v_2(b(v)) + O(1/n^3) \\ &= v + \frac{1}{n}B_1(v) + \frac{1}{n^2}B_2(v) + \frac{1}{n}v_1\left(v + \frac{1}{n}B_1(v)\right) + \frac{1}{n^2}v_2(v) + O(1/n^3) \\ &= v + \frac{1}{n}[B_1(v) + v_1(v)] + \frac{1}{n^2}[B_2(v) + B_1(v)v_1'(v) + v_2(v)] + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Balancing the $O(1/n)$ and $O(1/n^2)$ terms proves (15). \square

In the following we calculate an asymptotic expansion of the integral $\int_0^v F^n(x) dx$ using integration by parts (for an introduction to asymptotic evaluation of integrals using integration by parts, see, e.g., Murray (1984)):

Lemma 3 Let $F(v)$ be a twice-continuously differentiable, function and let $f = F' > 0$.

Then, for a sufficiently large n ,

$$\int_0^v F^n(x) dx = \frac{1}{n} \frac{F^{n+1}(v)}{f(v)} \left[1 + O\left(\frac{1}{n}\right) \right]. \quad (16)$$

Proof. Using integration by parts,

$$\begin{aligned} \int_0^v F^n(x) dx &= \int_0^v [F^n(x)f(x)] \frac{1}{f(x)} dx = \frac{1}{n+1} \frac{F^{n+1}(v)}{f(v)} + \frac{1}{n+1} \int_0^v [F^{n+1}(x)f(x)] \frac{f'(x)}{f^3(x)} dx \\ &= \frac{1}{n+1} \frac{F^{n+1}(v)}{f(v)} + \frac{1}{n+1} \frac{1}{n+2} F^{n+2}(v) \frac{f'(v)}{f^3(v)} - \frac{1}{n+1} \frac{1}{n+2} \int_0^v F^{n+2}(x) \left(\frac{f'(x)}{f^3(x)} \right)' dx. \end{aligned}$$

Therefore, the result follows.

B Proof of Proposition 1

Since $\lim_{n \rightarrow \infty} v(b) = b$, we can look for a solution of (2) of the form

$$v(b) = b + \frac{1}{n-1} v_1(b) + O\left(\frac{1}{n^2}\right).$$

Substitution in (2) gives

$$\begin{aligned} 1 + O\left(\frac{1}{n}\right) &= \frac{1}{n-1} \frac{F(b) + (v_1/(n-1))f(b) + O(n^{-2})}{f(b) + (v_1/(n-1))f'(b) + O(n^{-2})} \cdot \frac{U'(0) + (v_1/(n-1))U''(0) + O(n^{-2})}{U(0) + (v_1/(n-1))U'(0) + O(n^{-2})}. \end{aligned}$$

Since $U(0) = 0$ and $U'(0) > 0$, the balance of the leading order terms gives

$$1 = \frac{F(b)}{f(b)} \cdot \frac{U'(0)}{v_1 U'(0)}.$$

Therefore, $v_1(b) = F(b)/f(b)$ and the inverse equilibrium bids are given by

$$v(b) = b + \frac{1}{n-1} \frac{F(b)}{f(b)} + O\left(\frac{1}{n^2}\right).$$

Inverting this relation (see Lemma 2) shows that the equilibrium bids are given by (3).

To calculate the expected revenue, we use (3) to obtain

$$\begin{aligned} R[F] &= \int_0^1 b(v) dF^n(v) = b(1) - \int_0^1 b'(v) F^n(v) dv \\ &= 1 - \frac{1}{n} \frac{1}{f(1)} + O\left(\frac{1}{n^2}\right) - \int_0^1 [1 + O(1/n)] F^n(v) dv. \end{aligned}$$

Therefore, by (16), the result follows.

C Proof of Proposition 2

Since $\lim_{n \rightarrow \infty} v(b) = b$, we can look for a solution of the form

$$v(b) = b + \frac{1}{n-1} v_1(b) + \frac{1}{(n-1)^2} v_2(b) + O\left(\frac{1}{n^3}\right). \quad (17)$$

Substituting (17) in (2) and using $U(0) = 0$ and $0 < U'(0) < \infty$ gives

$$\begin{aligned} &1 + \frac{1}{n-1} v_1'(b) + O\left(\frac{1}{n^2}\right) \\ &= \frac{1}{n-1} \frac{F(b) + \frac{v_1}{n-1} f(b) + O(n^{-2})}{f(b) + \frac{v_1}{n-1} f'(b) + O(n^{-2})} \cdot \frac{U'(0) + (v_1/(n-1))U''(0) + O(n^{-2})}{U(0) + (\frac{v_1}{n-1} + \frac{v_2}{(n-1)^2})U'(0) + \frac{v_1^2}{2(n-1)^2}U''(0) + O(n^{-3})} \\ &= \frac{F(b) + \frac{v_1}{n-1} f(b) + O(n^{-2})}{f(b) + \frac{v_1}{n-1} f'(b) + O(n^{-2})} \cdot \frac{U'(0) + \frac{v_1}{n-1} U''(0) + O(n^{-2})}{(v_1 + \frac{v_2}{(n-1)})U'(0) + \frac{v_1^2}{2(n-1)}U''(0) + O(n^{-2})} \\ &= \left(\frac{F(b)}{f(b)} + \frac{v_1}{n-1} + O(n^{-2})\right) \left(1 - \frac{v_1}{n-1} \frac{f'(b)}{f(b)} + O(n^{-2})\right) \times \\ &\quad \left(\frac{1}{v_1} + \frac{1}{n-1} \frac{U''(0)}{U'(0)} + O(n^{-2})\right) \left(1 - \frac{1}{(n-1)} \frac{v_2}{v_1} - \frac{v_1}{2(n-1)} \frac{U''(0)}{U'(0)} + O(n^{-2})\right) \\ &= \frac{F(b)}{f(b)} \frac{1}{v_1} + \frac{1}{n-1} \left[1 - \frac{f'(b)F(b)}{f^2(b)} + \frac{F(b)}{2f(b)} \frac{U''(0)}{U'(0)} - \frac{F(b)}{f(b)} \frac{v_2}{v_1^2}\right] + O(n^{-2}). \end{aligned}$$

Balancing the $O(1)$ terms gives, as before,

$$v_1(b) = \frac{F(b)}{f(b)}. \quad (18)$$

Balancing the $O(\frac{1}{n-1})$ terms gives

$$v_1'(b) = 1 - \frac{f'(b)F(b)}{f^2(b)} + \frac{F(b)U''(0)}{2f(b)U'(0)} - \frac{F(b)v_2}{f(b)v_1^2}.$$

Substituting $v_1(b) = F(b)/f(b)$ and $v_1'(b) = 1 - \frac{F(b)f'(b)}{f^2(b)}$ gives

$$v_2(b) = \frac{F^2(b)U''(0)}{2f^2(b)U'(0)}. \quad (19)$$

Using Lemma 2 and (18,19) to invert the expansion (17) gives

$$b(v) = v + \frac{1}{n-1}B_1(v) + \frac{1}{(n-1)^2}B_2(v) + O\left(\frac{1}{n^3}\right),$$

where

$$B_1(v) = -\frac{F(v)}{f(v)}, \quad B_2(v) = \frac{F(v)}{f(v)} - \frac{F^2(v)f'(v)}{f^3(v)} - \frac{F^2(v)U''(0)}{2f^2(v)U'(0)}.$$

This completes the proof of (5).

To calculate the expected revenue, we first use (5) to obtain

$$\begin{aligned} R[F] &= \int_0^1 b(v) dF^n(v) = b(1) - \int_0^1 b'(v)F^n(v) dv \\ &= 1 - \frac{1}{n-1} \frac{1}{f(1)} + \frac{1}{(n-1)^2} \left[\frac{1}{f(1)} - \frac{f'(1)}{f^3(1)} - \frac{1}{2f^2(1)} \frac{U''(0)}{U'(0)} \right] \\ &\quad - \int_0^1 \left[1 - \frac{1}{n-1} \left(\frac{F(v)}{f(v)} \right)' \right] F^n(v) dv + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Integration by integration by parts (as in Lemma 3) gives,

$$\int_0^1 F^n(v) dv = \frac{1}{n+1} \frac{1}{f(1)} + \frac{1}{n+1} \frac{1}{n+2} \frac{f'(1)}{f^3(1)} + O\left(\frac{1}{n^3}\right),$$

and

$$\int_0^1 \left(\frac{F(v)}{f(v)} \right)' F^n(v) dv = \frac{1}{n+1} \frac{1}{f(1)} \left(\frac{F(v)}{f(v)} \right)'_{v=1} + O\left(\frac{1}{n^2}\right).$$

Therefore,

$$\begin{aligned}
& \int_0^1 \left[1 - \frac{1}{n-1} \left(\frac{F(v)}{f(v)} \right)' \right] F^n(v) dv \\
&= \frac{1}{n} \frac{1}{1 + \frac{1}{n}} \frac{1}{f(1)} + \frac{1}{n^2} \frac{f'(1)}{f^3(1)} - \frac{1}{n^2} \frac{1}{f(1)} \left(1 - \frac{f'(1)}{f^2(1)} \right) + O\left(\frac{1}{n^3}\right) \\
&= \frac{1}{n} \frac{1}{f(1)} - \frac{1}{n^2} \frac{2}{f(1)} + \frac{2}{n^2} \frac{f'(1)}{f^3(1)} + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

Substitution in the expression for $R[F]$ proves (6).

D Proof of Lemma 1

In the case of CRRA risk-aversion, equation (2) can be solved explicitly, yielding

$$b(v) = v - \frac{\int_0^v F^{\frac{n-1}{\alpha}}(s) ds}{F^{\frac{n-1}{\alpha}}(v)}.$$

Asymptotic expansion similar to the one in Lemma 3 gives

$$\int_0^v F^{\frac{n-1}{\alpha}}(s) ds = \frac{F^{\frac{n-1}{\alpha}+1}(v)}{\frac{n-1}{\alpha}+1} \frac{1}{f(v)} + O\left(\frac{1}{n^2}\right).$$

Therefore, equation (7) follows. The proof of (8) is similar to the proof of (4) in Proposition 1.

E Proof of Proposition 3

The case $k = 1$ was proved in Proposition 1. When $k = 2$ the result follows immediately, since $b(v) = v$. Therefore, we only need to prove for $k \geq 3$. In that case, the equilibrium strategies in k -price auctions are the solutions of (see Monderer and Tennenholtz (2000))

$$\int_0^v U(v - b(t)) F^{n-k}(t) (F(v) - F(t))^{k-3} f(t) dt = 0. \quad (20)$$

Defining $m = n - k$ and $t = v - s$, we can rewrite equation (20) as

$$\begin{aligned}
0 &= \int_0^v U(v - b(t)) F^m(t) (F(v) - F(t))^{k-3} f(t) dt \\
&= \int_0^v e^{m \ln(F(t))} U(v - b(t)) (F(v) - F(t))^{k-3} f(t) dt \\
&= e^{m \ln F(v)} \int_0^v e^{-m[\ln F(v) - \ln F(v-s)]} U(v - b(v-s)) (F(v) - F(v-s))^{k-3} f(v-s) ds.
\end{aligned} \tag{21}$$

Since the maximum of $\ln(F(v-s))$ is attained at $s = 0$, we can calculate an asymptotic approximation of this integral using Laplace method (see, e.g., Murray (1984)). To do that, we make the change of variables $x(s) = [\ln F(v) - \ln F(v-s)]$ and expand all the terms in the last integral in a Taylor series in s near $s = 0$.

Expansion of $x(s)$ near $s = 0$ gives $x = sf(v)/F(v) + O(s^2)$. Therefore,

$$\frac{dx}{ds} = f(v)/F(v) + O(s), \quad s = x \frac{F(v)}{f(v)} + O(s^2), \quad ds = \frac{dx}{f(v)/F(v)} [1 + O(x)]$$

Let us expand the solution $b(v)$ in a power series in m , i.e.,

$$b(v) = b_0(v) + \frac{1}{m} b_1(v) + O\left(\frac{1}{m^2}\right).$$

Therefore, near $s = 0$,

$$b(v-s) = b_0(v) - sb'_0(v) + \frac{1}{m} b_1(v) - \frac{1}{m} s b'_1(v) + O(s^2) + O\left(\frac{1}{m^2}\right).$$

In addition,

$$(F(v) - F(v-s))^{k-3} = (sf(v) + O(s^2))^{k-3} = s^{k-3} f^{k-3}(v) [1 + O(s)],$$

and

$$f(v-s) = f(v) + O(s).$$

Substitution all the above in (21) gives

$$\begin{aligned}
0 &= \int_0^v \left\{ e^{-mx} U \left[v - \left(b_0(v) - sb'_0(v) + \frac{b_1(v)}{m} - \frac{sb'_1(v)}{m} + O(s^2) + O\left(\frac{1}{m^2}\right) \right) \right] \times \right. \\
&\quad \left. s^{k-3} f^{k-3}(v) [1 + O(s)] [f(v) + O(s)] \right\} ds \\
&\sim \int_0^\infty \left\{ e^{-mx} U \left[v - \left(b_0(v) - x \frac{F(v)}{f(v)} b'_0(v) + \frac{1}{m} b_1(v) - \frac{x}{m} \frac{F(v)}{f(v)} b'_1(v) + O(x^2) + O\left(\frac{1}{m^2}\right) \right) \right] \times \right. \\
&\quad \left. x^{k-3} F^{k-3}(v) [1 + O(x)] [f(v) + O(x)] \frac{dx}{f(v)/F(v)} [1 + O(x)] \right\} \\
&= F^{k-2}(v) \int_0^\infty \left\{ e^{-mx} \left[U(v - b_0(v)) + U'(v - b_0(v)) \left(x \frac{F(v)}{f(v)} b'_0(v) - \frac{b_1(v)}{m} + \frac{x}{m} \frac{F(v)}{f(v)} b'_1(v) \right) \right. \right. \\
&\quad \left. \left. + O(x^2) + O\left(\frac{1}{m^2}\right) \right] x^{k-3} [1 + O(x)] \right\} dx. \tag{22}
\end{aligned}$$

We recall that for p integer, $\int_0^\infty e^{-mx} x^p dx = p!/m^{p+1}$. Therefore, balancing the leading $O(m^{-(k-2)})$ terms gives

$$U(v - b_0(v)) F^{k-2}(v) \int_0^\infty e^{-mx} x^{k-3} dx = 0.$$

Since $U(z) = 0$ only at $z = 0$, this implies that $b_0(v) \equiv v$. Using this and $U'(0) = 0$, equation (22) reduces to

$$0 = \int_0^\infty \left\{ e^{-mx} \left(x \frac{F(v)}{f(v)} - \frac{1}{m} b_1(v) + \frac{x}{m} \frac{F(v)}{f(v)} b'_1(v) \right) [x^{k-3} + O(x^{k-2})] \right\} dx$$

Therefore, balance of the next-order $O(m^{-(k-1)})$ terms gives

$$\frac{F(v)}{f(v)} \int_0^\infty e^{-mx} x^{k-2} dx - \frac{1}{m} b_1(v) \int_0^\infty e^{-mx} x^{k-3} dx = 0,$$

or

$$\frac{F(v)}{f(v)} \frac{(k-2)!}{m^{k-1}} - \frac{(k-3)!}{m^{k-1}} b_1(v) = 0.$$

Therefore,

$$b_1(v) = (k-2) \frac{F(v)}{f(v)}.$$

Hence, we proved (9).

The seller's expected revenue in a k -price auction is given by

$$R_k = \int_0^1 b(v) dF_k(v),$$

where $b(v)$ is the equilibrium bid in the k price auction and $F_k(v)$ is the distribution of the k -th valuation in order (i.e., k -order statistic of the bidders private valuations).

Substituting the asymptotic expansion for the equilibrium bids gives

$$R_k = \int_0^1 \left[v + \frac{k-2}{n-k} \frac{F(v)}{f(v)} \right] dF_k(v) + O\left(\frac{1}{n^2}\right).$$

Since the asymptotic expansion for the equilibrium bid is independent of the utility function U until order $O(\frac{1}{n^2})$, the revenue in the risk-averse case is the same as in the risk-neutral case, with $O(\frac{1}{n^2})$ accuracy. By the revenue equivalence theorem, the latter is given by (4).

F Proof of Proposition 4

From (11) it follows that

$$R^{2\text{nd}}[F_1, \dots, F_n] = 1 - \int_0^1 \prod_{i=1}^n F_i(v) dv - \sum_{i=1}^n \int_0^1 \frac{(1 - F_i(v))}{F_i(v)} \prod_{j=1}^n F_j(v) dv. \quad (23)$$

Repeating the proof of Lemma 3 with $F = F_G$ gives

$$\begin{aligned} \int_0^1 \prod_{i=1}^n F_i(v) dv &= \int_0^1 F_G^n(v) dv = \\ &= \frac{1}{n+1} \frac{1}{f_G(1)} - \frac{1}{n+1} \frac{1}{n+2} \frac{1}{f_G(1)} \left(\frac{1}{f_G} \right)' \Big|_{v=1} + \frac{1}{n+1} \frac{1}{n+2} \int_0^1 F_G^{n+2}(v) \left(\frac{1}{f_G(v)} \left(\frac{1}{f_G(v)} \right)' \right)' dv. \end{aligned}$$

Since $1/f_G(1)$, $(1/f_G)'|_{v=1}$ and $\int_0^1 F_G^{n+2}(v) \left(\frac{1}{f_G(v)} \left(\frac{1}{f_G(v)} \right)' \right)' dv$ are uniformly bounded in n , see Lemma 4,

$$\int_0^1 \prod_{i=1}^n F_i(v) dv = \frac{1}{n} \frac{1}{f_G(1)} + O\left(\frac{1}{n^2}\right). \quad (24)$$

In order to continue to evaluate the remaining terms in (23), we first use two integrations by parts

$$\begin{aligned} & \int_0^1 \frac{(1 - F_i(v))}{F_i(v)} \prod_{j=1}^n F_j(v) dv \\ &= \frac{1}{n+1} \frac{F_G^{n+1}(v)}{f_G(v)} \frac{1 - F_i(v)}{F_i(v)} \Big|_0^1 - \frac{1}{n+1} \int_0^1 F_G^{n+1}(v) \frac{f_G(v)}{f_G(v)} \left(\frac{1}{f_G(v)} \frac{1 - F_i(v)}{F_i(v)} \right)' dv \\ &= 0 - \frac{1}{n+1} \frac{1}{n+2} \frac{F_G^{n+2}(v)}{f_G(v)} \left(\frac{1}{f_G(v)} \frac{1 - F_i(v)}{F_i(v)} \right)' \Big|_0^1 \\ & \quad + \frac{1}{n+1} \frac{1}{n+2} \int_0^1 F_G^{n+2}(v) \left(\frac{1}{f_G(v)} \left(\frac{1}{f_G(v)} \frac{1 - F_i(v)}{F_i(v)} \right)' \right)' dv \\ &= \frac{1}{n^2} \frac{f_i(1)}{f_G^2(1)} + \frac{1}{n^2} \int_0^1 F_G^{n+2}(v) \left(\frac{1}{f_G(v)} \left(\frac{1}{f_G(v)} \frac{1 - F_i(v)}{F_i(v)} \right)' \right)' dv + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Arguments similar to the ones in Lemma 4 show that $F_G^3(v) \left(\frac{1}{f_G(v)} \left(\frac{1}{f_G(v)} \frac{1 - F_i(v)}{F_i(v)} \right)' \right)'$ is uniformly bounded. Therefore,

$$\left| \int_0^1 F_G^{n+2}(v) \left(\frac{1}{f_G(v)} \left(\frac{1}{f_G(v)} \frac{1 - F_i(v)}{F_i(v)} \right)' \right)' dv \right| \leq C \int_0^1 F_G^{n-1}(v) dv = O(1/n),$$

where is the last stage we used (24). Since, in addition, $f_G(1) = \frac{1}{n} \sum_{i=1}^n f_i(1)$,

$$\sum_{i=1}^n \int_0^1 \frac{(1 - F_i(v))}{F_i(v)} \prod_{j=1}^n F_j(v) dv = \frac{1}{n^2} \frac{\sum_{i=1}^n f_i(1)}{f_G^2(1)} + O\left(\frac{1}{n^2}\right) = \frac{1}{n} \frac{1}{f_G(1)} + O\left(\frac{1}{n^2}\right).$$

Substituting the last relation and (24) in (23) concludes the proof.

Lemma 4 *There exists a positive number C , independent of n , such that*

$$|1/f_G(1)| \leq C, \quad |(1/f_G)'(1)| \leq C, \quad \left| \int_0^1 F_G^{n+2}(v) \left(\frac{1}{f_G(v)} \left(\frac{1}{f_G(v)} \right)' \right)' dv \right| \leq C.$$

Proof: From Assumptions (10) it follows that $mv \leq F_j(v) \leq Mv$. Therefore, $mv \leq F_G(v) \leq Mv$, and

$$\frac{m}{M} \leq \frac{F_G(v)}{F_j(v)} \leq \frac{M}{m}, \quad 0 \leq v \leq 1. \quad (25)$$

Since

$$f_G(v) = \frac{1}{n} F_G(v) \sum_{j=1}^n \frac{f_j(v)}{F_j(v)} = \frac{1}{n} \sum_{j=1}^n \frac{F_G(v)}{F_j(v)} f_j(v), \quad (26)$$

equation (25) implies that

$$\frac{m^2}{M} \leq f_G(v) \leq \frac{M^2}{m}, \quad \frac{m}{M^2} \leq \frac{1}{f_G(v)} \leq \frac{M}{m^2}. \quad (27)$$

In particular, we see that $1/f_G(1)$ is uniformly bounded.

Differentiating equation (26) gives

$$f'_G(v) = \left[\frac{1}{n^2} F_G(v) \left(\sum_{j=1}^n \frac{f_j(v)}{F_j(v)} \right)^2 + \frac{1}{n} F_G(v) \sum_{j=1}^n \frac{f'_j(v) F_j(v) - f_j^2(v)}{F_j^2(v)} \right]. \quad (28)$$

Therefore, from assumptions (10) it immediately follows that $|f'_G(1)|$ is uniformly bounded, hence also $|(1/f_G)'|_{v=1}$.

In order to bound the integral, we first note that since $F_G^n(v) \leq 1$, it is enough to show that

$$F_G^2(v) \left(\frac{1}{f_G(v)} \left(\frac{1}{f_G(v)} \right)' \right)' = -\frac{F_G^2(v) f_G''(v)}{f_G^3(v)} + 3 \frac{F_G^2(v) (f'_G(v))^2}{f_G^4(v)}$$

is bounded (uniformly in v and in n). In light of (27), we only need to show that $F_G^2 f_G''$ and $F_G f'_G$ are bounded. Indeed, by (28),

$$F_G(v) f'_G(v) = \frac{1}{n^2} \left(\sum_{j=1}^n \frac{F_G(v)}{F_j(v)} f_j(v) \right)^2 + \frac{1}{n} \sum_{j=1}^n \frac{F_G^2(v)}{F_j^2(v)} (f'_j(v) F_j(v) - f_j^2(v)).$$

Therefore, relations (10,27) show that $|F_G(v)f'_G(v)|$ is uniformly bounded. Similar arguments show that

$$\begin{aligned}
F_G^2(v)f''_G(v) &= \frac{1}{n^3} \left(\sum_{j=1}^n \frac{F_G(v)}{F_j(v)} f_j(v) \right)^3 + \frac{3}{n^2} \sum_{j=1}^n \frac{F_G(v)f_j(v)}{F_j(v)} \sum_{j=1}^n \frac{F_G^2(v)}{F_j^2(v)} (f'_j(v)F_j(v) - f_j^2(v)) \\
&\quad + \frac{1}{n} F_G(v) \sum_{j=1}^n \frac{F_G^2(v)}{F_j^2(v)} (f''_j(v)F_j(v) - f_j(v)f'_j(v)) \\
&\quad - \frac{1}{n} \sum_{j=1}^n \frac{F_G^3(v)}{F_j^3(v)} 2 (f'_j(v)F_j(v) - f_j^2(v)) f_j(v)
\end{aligned}$$

is uniformly bounded.

G Proof of Theorem 1

When all players with a zero valuation expect a zero surplus (Condition 4), then the seller's expected revenue is given by (Fibich, Gaviols and Sela, 2003)

$$R = - \sum_{i=1}^n \int_{\underline{v}}^{\bar{v}} P_i(v) [v(1 - F_i(v))] ' dv, \quad (29)$$

where $P_i(v)$ is the probability of winning for bidder i with type v at equilibrium. Since

$$P_i(v) = P \left(b_i(v) > \max_{j \neq i} b_j \right) = \prod_{\substack{j=1 \\ j \neq i}}^n F_j(b_j^{-1}(b_i(v))),$$

we can rewrite (29) as

$$R = - \sum_{i=1}^n \int_0^1 \left(\prod_{j=1}^n F_j(b_j^{-1}(b_i(v))) \right) \frac{1 - F_i(v) - v f_i(v)}{F_i(v)} dv.$$

In order to approximate R for large n , we use an asymptotic method known as Laplace method for integrals (see, e.g., Murray (1984)). We first define

$$G_i(v) = \left(\prod_{j=1}^n F_j(b_j^{-1}(b_i(v))) \right)^{1/n}, \quad g_i(v) = \frac{d}{dv} G_i(v).$$

Then, we have

$$\begin{aligned}
R &= - \sum_{i=1}^n \int_0^1 e^{n \ln G_i(v)} \cdot \left(\frac{1 - F_i(v) - v f_i(v)}{F_i(v)} \right) dv \\
&= - \sum_{i=1}^n \int_0^1 e^{n \ln G_i(1-s)} \cdot \left(\frac{1 - F_i(1-s) - (1-s) f_i(1-s)}{F_i(1-s)} \right) ds, \quad (30)
\end{aligned}$$

where the last equality is obtained by the change of variables $s = 1 - v$.

The essence of Laplace method is the realization that as n increases, the contribution to each integral comes essentially only from the neighborhood of the point where G_i attains its maximum, i.e., near $v = 1$ or $s = 0$. Hence, with exponential accuracy, one can replace the integrand with its Taylor expansion near $s = 0$, and also extend the domain of integration from $[0, 1]$ to $[0, \infty)$. To do that, let us first prove the following result:

Lemma 5 *Let $x_i(s) = -\ln G_i(1-s)$, and let $0 \leq s \ll 1$. Then,*

$$s = \frac{x_i}{g_i(1)} (1 + O(x_i)), \quad ds = \frac{dx_i}{g_i(1)} \left[1 + x_i \left(\frac{g_i'(1)}{g_i^2(1)} - 1 \right) + O(x_i^2) \right].$$

Proof. Taylor expansion in s gives¹³

$$x_i = s g_i(1) + \frac{s^2}{2} \eta(z(s)), \quad \eta(s) = \frac{d}{ds} \left(\frac{g_i(1-s)}{G_i(1-s)} \right), \quad 0 \leq z(s) \leq s.$$

Solving the quadratic equation for s (for the root that satisfies $x(s=0) = 0$) gives

$$s = \frac{-g_i(1) + \sqrt{g_i^2(1) + 2x_i\eta}}{\eta},$$

Since $x_i \ll 1$, and since $|\eta(z(s))|, |1/g_i(1)| \leq C$, see Lemma 6, we can expand this expression as

$$s = \frac{-g_i(1) + g_i \sqrt{1 + 2x_i\eta/g_i^2(1)}}{\eta} = \frac{x_i}{g_i(1)} \left(1 - \frac{x_i\eta}{2g_i^2(1)} + \dots \right) = \frac{x_i}{g_i(1)} (1 + O(x_i)).$$

¹³Here we used Condition 5 to deduce that $G_i(1) = 1$.

Similarly,

$$\frac{ds}{dx_i} = \frac{G_i(1-s)}{g_i(1-s)} = \frac{1}{g_i(1)} - s \frac{g_i^2(1) - g_i'(1)}{g_i^2(1)} + \frac{s^2}{2} \xi(w(s)), \quad \xi = \frac{d^2}{ds^2} \left(\frac{G_i(1-s)}{g_i(1-s)} \right), \quad 0 \leq w(s) \leq s.$$

Since $|\xi(w(s))| \leq C$, see Lemma 6,

$$ds = \frac{dx_i}{g_i(1)} \left[1 + x_i \left(\frac{g_i'(1)}{g_i^2(1)} - 1 \right) + O(x_i^2) \right].$$

□

In addition,

$$\begin{aligned} \frac{1 - F_i(1-s) - (1-s)f_i(1-s)}{F_i(1-s)} &= -f_i(1) - s(f_i^2(1) - f_i'(1) - 2f_i(1)) + O(s^2) \\ &= -f_i(1) - \frac{x_i}{g_i(1)}(f_i^2(1) - f_i'(1) - 2f_i(1)) + O(x_i^2). \end{aligned}$$

Substituting the above in (30) gives that

$$R = \int_0^\infty \sum_{i=1}^n \left[e^{-nx_i} \left(f_i(1) + \frac{x_i}{g_i(1)}(f_i^2(1) - f_i'(1) - 2f_i(1)) \right) \left(1 + x_i \left(\frac{g_i'(1)}{g_i^2(1)} - 1 \right) \right) + O(x_i^2) \right] \frac{dx_i}{g_i(1)}.$$

Since x_i is a dummy variable, we can rename $x_i = x$. Therefore,

$$\begin{aligned} R &= \sum_{i=1}^n \int_0^\infty \left\{ e^{-nx} \cdot \left(f_i(1) + \frac{x}{g_i(1)}(f_i^2(1) - f_i'(1) - 2f_i(1)) \right) \left(1 + x \left(\frac{g_i'(1)}{g_i^2(1)} - 1 \right) \right) + O(x^2) \right\} \frac{dx}{g_i(1)} \\ &= \sum_{i=1}^n \frac{1}{g_i(1)} \int_0^\infty e^{-nx} \left(f_i(1) + x \left[\frac{1}{g_i(1)}(f_i^2(1) - f_i'(1) - 2f_i(1)) + f_i(1) \left(\frac{g_i'(1)}{g_i^2(1)} - 1 \right) \right] + O(x^2) \right) dx \\ &= \sum_{i=1}^n \frac{f_i(1)}{g_i(1)} \int_0^\infty e^{-nx} dx \\ &\quad + \sum_{i=1}^n \frac{1}{g_i(1)} \left[\frac{1}{g_i(1)}(f_i^2(1) - f_i'(1) - 2f_i(1)) + f_i(1) \left(\frac{g_i'(1)}{g_i^2(1)} - 1 \right) \right] \int_0^\infty e^{-nx} x dx + O(n^{-2}). \end{aligned}$$

We thus see that

$$\begin{aligned}
R &= \frac{1}{n} \sum_{i=1}^n \frac{f_i(1)}{g_i(1)} + \frac{1}{n^2} \sum_{i=1}^n \frac{1}{g_i^2(1)} \left[(f_i^2(1) - f_i'(1) - 2f_i(1)) + f_i(1) \frac{g_i'(1)}{g_i(1)} \right] \\
&\quad - \frac{1}{n^2} \sum_{i=1}^n \frac{f_i(1)}{g_i(1)} + O(n^{-2}). \tag{31}
\end{aligned}$$

In order to calculate the first term on the right-hand-side of (31), we first note that

$$\begin{aligned}
g_i(v) &= \frac{d}{dv} \left(\prod_{j=1}^n F_j(b_j^{-1}(b_i(v))) \right)^{1/n} \\
&= \frac{1}{n} \left(\prod_{k=1}^n F_k(b_k^{-1}(b_i(v))) \right)^{1/n-1} \sum_{j=1}^n \left(\prod_{k \neq j}^n F_k(b_k^{-1}(b_i(v))) \right) f_j(b_j^{-1}(b_i(v))) \frac{b_i'(v)}{b_j'(b_j^{-1}(b_i(v)))} \\
&= G_i(v) \frac{1}{n} b_i'(v) \sum_{j=1}^n \frac{f_j(b_j^{-1}(b_i(v)))}{F_j(b_j^{-1}(b_i(v)))} \frac{1}{b_j'(b_j^{-1}(b_i(v)))}. \tag{32}
\end{aligned}$$

Since all bidders have the same maximal bid, then $b_j^{-1}(b_i(1)) = 1$. Therefore,

$$g_i(1) = \frac{b_i'(1)}{n} \sum_{j=1}^n \frac{f_j(1)}{b_j'(1)}. \tag{33}$$

From this, it follows that

$$g_i(1) = \alpha b_i'(1), \tag{34}$$

where α is a constant which is independent of i , and hence that

$$\frac{1}{n} \sum_{i=1}^n \frac{f_i(1)}{g_i(1)} = 1. \tag{35}$$

Differentiating (32) and using $b_j^{-1}(b_i(1)) = 1$ gives

$$\begin{aligned}
g_i'(1) &= \frac{1}{n^2} (b_i'(1))^2 \left(\sum_{j=1}^n \frac{f_j(1)}{b_j'(1)} \right)^2 + \frac{1}{n} b_i''(1) \sum_{j=1}^n \frac{f_j(1)}{b_j'(1)} \\
&\quad + \frac{1}{n} b_i'(1) \sum_{j=1}^n \left(\frac{f_j'(1)}{b_j'(1)} - \frac{f_j^2(1)}{b_j'(1)} - \frac{f_j(1)b_j''(1)}{(b_j'(1))^2} \right) \frac{b_i'(1)}{b_j'(1)} \\
&= g_i^2(1) + \frac{b_i''(1)}{b_i'(1)} g_i(1) + \frac{1}{n} b_i'(1) \sum_{j=1}^n \left(\frac{f_j'(1)}{b_j'(1)} - \frac{f_j^2(1)}{b_j'(1)} - \frac{f_j(1)b_j''(1)}{(b_j'(1))^2} \right) \frac{b_i'(1)}{b_j'(1)}. \tag{36}
\end{aligned}$$

Multiplying by $f_i(1)$ and using (34) gives

$$f_i(1)g_i'(1) = f_i(1)g_i^2(1) + \alpha b_i''(1)f_i(1) + \frac{1}{n}f_i(1)g_i^2(1) \sum_{j=1}^n \left(\frac{f_j'(1)}{g_j^2(1)} - \frac{f_j^2(1)}{g_j^2(1)} - \frac{\alpha f_j(1)b_j''(1)}{g_j^3(1)} \right).$$

Dividing by $g_i^3(1)$ and summing gives

$$\sum_{i=1}^n \left(\frac{f_i(1)g_i'(1)}{g_i^3(1)} \right) = \underbrace{\sum_{i=1}^n \frac{f_i(1)}{g_i(1)}}_{=n} + \alpha \sum_{i=1}^n \frac{b_i''(1)f_i(1)}{g_i^3(1)} + \underbrace{\left(\frac{1}{n} \sum_{i=1}^n \frac{f_i(1)}{g_i(1)} \right)}_{=1} \sum_{j=1}^n \left(\frac{f_j'(1)}{g_j^2(1)} - \frac{f_j^2(1)}{g_j^2(1)} - \frac{\alpha f_j(1)b_j''(1)}{g_j^3(1)} \right).$$

Therefore,

$$\sum_{i=1}^n \left(\frac{f_i(1)g_i'(1)}{g_i^3(1)} + \frac{f_i^2(1) - f_i'(1)}{g_i^2(1)} \right) = n.$$

Substituting this and (35) in (31) gives

$$R = 1 - \frac{2}{n^2} \sum_{i=1}^n \frac{f_i(1)}{g_i^2(1)} + O(n^{-2}). \quad (37)$$

Using (33) and Condition 6 gives that

$$g_i(1) = \frac{1}{n} \sum_{j=1}^n f_j(1) + O(1/n). \quad (38)$$

Substitution in (37) gives (14).

Lemma 6 *Let $s \ll 1$. Then, there exist a positive constant C , independent of n , such that*

$$|1/g_i(1)| \leq C, \quad |\eta(z(s))| \leq C, \quad |\xi(w(s))| \leq C.$$

Proof. From (33) and Assumptions ?? it follows that

$$m + O(1/n) \leq |g_i(1)| \leq M + O(1/n). \quad (39)$$

Since $s \ll 1$, it follows that $z(s), w(s) \ll 1$ and thus, it suffices to bound $|\eta(t)|$ and $|\xi(t)|$

for $t \ll 1$. Now,

$$\begin{aligned}\eta(t) &= \frac{-g'_i(1-t)G_i(1-t) + g_i^2(1-t)}{G_i^2(1-t)}, \\ \xi(t) &= \frac{-g_i^3(1-t)g'_i(1-t) - g_i''(1-t)g_i^2(1-t)G_i(1-t) + 2g_i(1-t)(g'_i)^2(1-t)G_i(1-t)}{g_i^4(1-t)}.\end{aligned}$$

Therefore, we need to bound $G_i(1-t)$, $g_i(1-t)$, $g'_i(1-t)$, $g''_i(1-t)$, $1/G_i(1-t)$ and $1/g_i(1-t)$. Since $G_i(1) = 1$, and since $g_i(1)$ is bounded from above and away from zero from below, see (39), it is enough to bound $g'_i(1)$ and $g''_i(1)$, since by continuity, the bound is still valid for $t \ll 1$. Using (36), and the conditions $b'_i(1)/b'_j(1) = 1 + O(1/n)$ and $b''_i(1)/b''_j(1) = 1 + O(1/n)$, gives that

$$g'_i(1) = g_i^2(1) + \frac{1}{n} \sum_{j=1}^n (f'_j(1) - f_j^2(1)) + O(1/n). \quad (40)$$

Similarly, using also the condition that $b'''_i(1)/b'''_j(1) = 1 + O(1/n)$ gives

$$\begin{aligned}g''_i(1) &= g_i^3(1) + \frac{3}{n}g_i(1) \sum_{j=1}^n (f'_j(1) - f_j^2(1)) + \\ &\quad + \frac{1}{n} \sum_{j=1}^n (f''_j(1) - 2f_j(1)f'_j(1) + f_j^3(1)) + O(1/n).\end{aligned}$$

Therefore, Assumptions (10) imply that $g'_i(1)$ and $g''_i(1)$ are uniformly bounded.

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