Common-Value All-Pay Auctions with Asymmetric Information

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Abstract

We study two-player common-value all-pay auctions in which the players have ex-ante asymmetric information represented by finite partitions of the set of possible values of winning. We consider a family of such auctions in which no player has an information advantage over his opponent. We find sufficient conditions for the existence of equilibrium in monotonic strategies. We further show that the ex-ante distribution of equilibrium effort is the same for every player (and hence the players' expected efforts are equal), although their expected payoffs are different and they do not have the same ex-ante probability of winning.

Keywords*:* Common-value all-pay auctions, asymmetric information.

JEL Classification: C72*,* D44, D82.

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1 Introduction

All-pay auctions are used in diverse areas of economics, such as lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, and biological wars of attrition. In an all-pay auction each player submits a "bid" (i.e., exerts effort) and the player with the highest bid wins the contest. However, regardless of who the winner is, each player bears the cost of his bid. All-pay auctions have been studied when players have either private or common values of winning.[1] In this paper we focus on common-value all-pay auctions and consider contests with two ex-ante *asymmetrically* informed players where the value of winning is identical for both players in the same state of nature, but the information about which state of nature was realized is different. We assume that each player's information is represented by a finite partition of the set of states of nature that can be identified with the set of possible common values, but these partitions are different.[2] When the state of nature is chosen, each player learns which element of partition contains the realized common-value, but the players do not necessarily know the exact value of winning the contest.[3] This model captures situations in which winning a contest is of similar benefit to each contestant, but the precise value of winning, which depends on several random parameters, may be unknown.

In our model of asymmetric information we assume that the information sets of each player are connected with respect to the value of winning the contest (see Einy et al. (2001, 2002) and Forges and Orzach (2011)). This means that if a player's information partition does not enable him to distinguish between two possible values of winning, then he also cannot distinguish between all intermediate values. Connectness seems plausible in environments where the information of a player allows him to put upper and lower bounds on the actual value of winning, without ruling out any outcome within these bounds.

In studying common-value all-pay auctions with asymmetric information, one particular case of interest

[1] To mention just a few works, all-pay auctions have been considered by, e.g., Hillman and Riley (1989), Baye et al. (1993, 1996), Amann and Leininger (1996), Che and Gale (1998), Moldovanu and Sela (2001, 2006), Siegel (2009) and Moldovanu et al. (2010).

[2] This partition representation is equivalent to the more common Harsanyi-type formulation of Bayesian games (see Jackson (1993) and Vohra (1999))

[3] This framework has been used in several works to analyze common-value second-price auctions (see Einy et al. (2001, 2002), Forges and Orzach (2011), and Abraham et al. (2012)), and common-value first-price auctions (see Malueg and Orzach (2009, 2012)).

is when one player has an information advantage (henceforth, IA) over another, which means that his information partition is finer than that of his opponent. This assumption simplifies the analysis when there are only two players, since it can usually be reduced to the postulate that one player is completely informed about the state of nature, while the other player has no information about it. Significant results have been obtained in the IA case. Siegel (2014) has shown[4] the existence of a unique equilibrium in mixed strategies in a two-player common value auction with IA, in which the ex-ante distribution of effort is the same for both players, as is the ex-ante probability of winning the contest.

In this paper we study two-player common-value all-pay auctions in which, except in the extreme states of nature (corresponding to the lowest and the highest possible values of winning), neither player has an IA over his opponent.[5] We construct a "candidate" for an equilibrium with monotonic strategies (i.e., strategies where more favorable signals do not lead to lower bids), and then find sufficient conditions for the "candidate" to be a true equilibrium.

Our results show that, although the players have asymmetric strategies that yield different expected payoffs as well as different chances to win the contest – unlike in the IA case, – the ex ante distributions of both players' efforts are the same – in common with the IA case. Thus the asymmetry of information between the players manifests itself not in unequal expected efforts, but in their chances to win the contest and the allocation of payoffs between the players (since in our model no player has an inherent advantage over his rival, the expected payoff of a player may be higher or lower than that of his opponent[6]). We also show that a player's expected payoff, conditional on an element of his partition, monotonically increases with the values of winning. Thus, higher/lower values in the information set of a player lead to a higher/lower conditional expected payoff.

As was already mentioned, our work is related to the recent research of Siegel (2014), who studies general asymmetric two-player all-pay auctions with interdependent valuations, where private information of each player is represented by a finite set of possible types. He shows that a unique equilibrium exists in his set-up, and provides an algorithm to calculate the equilibrium strategies. Siegel makes strong assumption

[4] See Section 2 of the online appendix to his work.

[5] Malueg and Orzach (2009) studied the first-price auction with information partitions of this type.

[6] As Siegel (2014) has shown, in the unique equilibrium of the IA case, the expected payoff of the uninformed player is zero, while the expected payoff of the informed payer is positive.

("Condition M") of joint monotonicity of conditional densities and valuations, that presupposes the common prior distribution of type profiles having full support. Thus, given his information, a player cannot rule out any specific type of his rival. In contrast, in our model, given one player's information there is always some information set of the other player which is ruled out.

The paper is organized as follows. In Section 2 we present the model. In Section 3 we give a numerical example that demonstrates how to find an equilibrium in our model. In Section 4 we find sufficient conditions for the existence of equilibrium and explicitly describe such an equilibrium. In Section 5 we analyze the players' expected efforts and payoffs. Section 6 concludes. Some of the proofs are in the Appendix.

2 The model

Consider the set $\mathcal{N} = \{1, 2\}$ of two players who compete in an all-pay auction where the player with the highest effort (output) wins the contest, but all the players bear the cost of their effort. The uncertainty in our model is described by a finite set Ω of states of nature and a probability distribution p over Ω which can be interpreted as the common prior belief about the realized state of nature (w.l.o.g. $p(\omega) > 0$ for every $\omega \in \Omega$). A function $v : \Omega \to \mathbb{R}_+$ represents the common value of winning the contest, i.e., if $\omega \in \Omega$ is realized then the value of winning is $v(\omega)$ for every player. The private information of each player $i \in \mathcal{N}$ is described by a partition Π_i of Ω.

A common-value all-pay auction starts when nature chooses a state ω form Ω according to the distribution p. Each player i is informed of the element $\pi_i(\omega)$ of Π_i which contains ω (thus, $\pi_i(\omega)$ constitutes the information set of player i at ω), and then he chooses an effort $x_i \in \mathbb{R}_+$. The players will typically have different information partitions, and thus are ex-ante asymmetric.

The utility (payoff) of player $i \in \mathcal{N}$ is given by the function $u_i : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}$ as follows:

$$u_i(\omega, x) = \begin{cases} \frac{1}{m(x)} v(\omega) - x_i, & \text{if} \quad x_i = \max\{x_1, x_2\}, \\ -x_i, & \text{if} \quad x_i < \max\{x_1, x_2\}, \end{cases}$$

where $m(x)$ denotes the number of players who exert the highest effort, namely, $m(x) = |i \in N : x_i = \max\{x_1, x_2\}|$. A two-player common-value all-pay auction with differential information is fully described by and identified with the collection $G = ((\Omega, p), u_1, u_2, \Pi_1, \Pi_2)$.

In all-pay auctions, there is usually no equilibrium in pure strategies. Thus our attention will be given to mixed strategy equilibria. A mixed strategy of player i is a function $F_i : \Omega \times \mathbb{R}_+ \to [0,1]$, such that for every $\omega \in \Omega$, $F_i(,\cdot)$ is a cumulative distribution function (c.d.f.) on $\mathbb{R}_+$, and for all $x \in \mathbb{R}_+$, $F_i(\cdot, x)$ is a Π_i-measurable function (that is, $F_i(\cdot, x)$ is constant on every element of Π_i). Slightly abusing notation, for any $\pi_i \in \Pi_i$ we will denote the constant value of $F_i(\cdot, x)$ on π_i by $F_i(\pi_i, x)$, whenever convenient. If player i plays a pure strategy given π_i, i.e., if the distribution represented by $F_i(\pi_i, \cdot)$ is supported on some $y \in \mathbb{R}_+$, we will identify between $F_i(\pi_i, \cdot)$ and y wherever appropriate.

Given a mixed strategy profile $F = (F_1, F_2)$, denote by $E_i(F)$ the expected payoff of player i when players use that strategy profile, i.e.,

$$E_i(F) \equiv E(\int_0^\infty \int_0^\infty u_i(\cdot, (x_1, x_2)) dF_1(\cdot, x_1) dF_2(\cdot, x_2)).$$

For $\pi_i \in \Pi_i$, $E_i(\pi_i, F)$ will denote the conditional expected payoff of player i given his information set π_i, i.e.,

$$E_i(\pi_i, F) \equiv E(\left[\int_0^\infty \int_0^\infty u_i(\cdot, (x_1, x_2)) dF_1(\cdot, x_1) dF_2(\cdot, x_2)\right] \mid \pi_i).$$

A profile $F^* = (F_1^*, F_2^*)$ of mixed strategies constitutes a (Bayesian Nash) equilibrium in the common-value all-pay auction G if for every player i, and every mixed strategy F_i of that player, the following inequality holds:

$$E_i(F^*) \geq E_i(F_i, F_{-i}^*),$$

where $-i$ denotes i's rival.

3 Example

We begin with a simple example of the behavior of players in our model. Consider a two-player common-value all-pay auction with three states of nature. For $i = 1, 2, 3$, state ω_i occurs with probability $p_i = \frac{1}{3}$ and the value for the prize in ω_i is $v(\omega_i) = i$. Player 1's information partition is $\Pi_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, while player 2's information partition is $\Pi_2 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$.

It can be easily verified that the corresponding common-value all-pay auction does not have an equilibrium in pure strategies. However, there does exist a mixed strategy equilibrium. In this equilibrium, player 1's

mixed strategy F_1^{**} is a state-dependent c.d.f. given by

$$F_1^{**}(\{\omega_1\}, x) = \begin{cases} 0, & \text{if } x < 0, \\ 2x, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1, & \text{if } x > \frac{1}{2} \end{cases}$$

and

$$F_1^{**}(\{\omega_2, \omega_3\}, x) = \begin{cases} 0, & \text{if } x < \frac{1}{2}, \\ x - \frac{1}{2}\ , & \text{if } \frac{1}{2} \leq x \leq 1, \\ \frac{x}{3} + \frac{1}{6}, & \text{if } 1 < x \leq \frac{5}{2}, \\ 1, & \text{if } x > \frac{5}{2}. \end{cases}$$

Player 2's mixed strategy F_2^{**} is a state-dependent c.d.f. given by

$$F_2^{**}(\{\omega_1, \omega_2\}, x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1 \end{cases}$$

and

$$F_2^{**}(\{\omega_3\}, x) = \begin{cases} 0, & \text{if } x < 1, \\ \frac{2}{3}x - \frac{2}{3}, & \text{if } 1 \leq x \leq \frac{5}{2}, \\ 1, & \text{if } x > \frac{5}{2}. \end{cases}$$

In order to see that the above strategies are an equilibrium, note that when player 2 uses mixed strategy F_2^{**}, player 1's expected payoff conditional on the event $\{\omega_1\}$ is

$$E_1(\{\omega_1\}, x, F_2^{**}) = 1 \cdot x - x = 0$$

for any effort $x \in [0, \frac{1}{2}]$. It is easy to see that efforts above $\frac{1}{2}$ would result in a non-positive conditional expected payoff to player 1. Thus, any effort in $[0, \frac{1}{2}]$ is a best response of player 1 to F_2^{**} conditional on $\{\omega_1\}$. Furthermore, player 1's expected payoff conditional on the event $\{\omega_2, \omega_3\}$ is as follows: when 1 exerts effort $x \in (1, \frac{5}{2}]$,

$$E_1(\{\omega_2, \omega_3\}, x, F_2^{**}) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 3 \cdot (\frac{2}{3}x - \frac{2}{3}) - x = 0,$$

and when 1 exerts effort $x \in [\frac{1}{2}, 1]$,

$$E_1(\{\omega_2, \omega_3\}, x, F_2^{**}) = \frac{1}{2} \cdot 2 \cdot x - x = 0.$$

It is easy to see that exerting efforts below $\frac{1}{2}$ or above $\frac{5}{2}$ would lead to non-positive conditional expected payoffs, and thus any effort in $[\frac{1}{2}, \frac{5}{2}]$ is a best response of player 1 to F_2^{**} conditional on $\{\omega_2, \omega_3\}$. Thus the mixed strategy F_1^{**} (supported on $[0, \frac{1}{2}]$ given $\{\omega_1\}$ and on $[\frac{1}{2}, \frac{5}{2}]$ given $\{\omega_2, \omega_3\}$) is an (unconditional) best response of player 1 to F_2^{**}.

Similarly, when payer 1 uses mixed strategy F_1^{**}, player 2's expected payoff conditional on the event $\{\omega_1, \omega_2\}$ payoff is as follows: when 2 exerts effort $x \in (\frac{1}{2}, 1]$,

$$E_2(\{\omega_1, \omega_2\}, F_1^{**}, x) = \frac{1}{2} + \frac{1}{2} \cdot 2 \cdot (x - \frac{1}{2}) - x = 0,$$

and when 1 exerts effort $x \in [0, \frac{1}{2}]$,

$$E_2(\{\omega_1, \omega_2\}, F_1^{**}, x) = \frac{1}{2} \cdot 1 \cdot 2x - x = 0.$$

It is easy to see that any effort above 1 would result in a negative expected payoff. Also, conditional on the event $\{\omega_3\}$, the expected payoff of player 2 when he exerts effort $x \in [1, \frac{5}{2}]$ is

$$E_2(\{\omega_3\}, x), F_1^{**}, x) = 3 \cdot (\frac{x}{3} + \frac{1}{6}) - x = \frac{1}{2}.$$

Any effort above $\frac{5}{2}$ or below 1 would lead to a conditional expected payoff smaller than $\frac{1}{2}$. Thus the mixed strategy F_2^{**} (supported on $[0, 1]$ given $\{\omega_1, \omega_2\}$ and on $[1, \frac{5}{2}]$ given $\{\omega_3\}$) is an (unconditional) best response of player 2 to F_1^{**}.

Hence, the pair $F^{**} = (F_1^{**}, F_2^{**})$ is a mixed strategy equilibrium. The ex-ante distributions of each player's equilibrium effort are *identical* – indeed, the ex-ante probability to exert an effort smaller than or equal to x is the same for both players and is given by

$$\overline{F}^{**}(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{2}{3}x\ , & \text{if } 0 \le x \le 1, \\ \frac{2x}{9} + \frac{4}{9}, & \text{if } 1 < x \le \frac{5}{2}, \\ 1, & \text{if } x > \frac{5}{2}. \end{cases}$$

In particular, the expected equilibrium efforts of both players are identical and equal to 2.4722. The expected payoff of the players in F^{**} are

$$\begin{aligned} E_1(F_1^{**}, F_2^{**}) &= 0, \\ E_2(F_1^{**}, F_2^{**}) &= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}. \end{aligned}$$

The ex-ante probability of player 2 to win is given by

$$\begin{aligned} P_2 &= \frac{1}{3}[\int_0^{\frac{1}{2}} \left(\int_0^x 2ds \right) 1dx + \int_{\frac{1}{2}}^1 1dx] \\ &\quad + \frac{1}{3} \int_{\frac{1}{2}}^1 \left(\int_{\frac{1}{2}}^x 1ds \right) 1dx \\ &\quad + \frac{1}{3}[\int_1^{\frac{5}{2}} \left(\int_1^x \frac{1}{3} ds \right) \frac{2}{3} dx + \int_{\frac{1}{2}}^1 1ds] \\ &= \frac{13}{24} \end{aligned}$$

Thus, the ex-ante probabilities of players 1 and 2 to win are *distinct*, $\frac{11}{24}$ and $\frac{13}{24}$ respectively.

In the next section we will present sufficient conditions for the existence of a mixed-strategy equilibrium, and describe the monotonic equilibrium strategies in a general two-player common-value all-pay auction without information advantage.

4 Equilibrium analysis

Assume that $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$, where $n > 1$ is without loss of generality an odd number. For each state of nature $\omega_i \in \Omega$, denote

$$v_i = v(\omega_i) \text{ and } p_i = p(\omega_i) > 0, \tag{1}$$

and assume that that the possible values are positive and ranked as follows:

$$0 < v_1 \leq v_2 \leq ... \leq v_n. \tag{2}$$

Also consider:

Assumption1: Each partition $\Pi_i, i = 1, 2$ is *connected* with respect to the common value function v, i.e., for every element $\pi_j \in \Pi_j$, if $\omega_i, \omega_{i+k} \in \pi_j$ and $k > 1$, then also $\omega_{i+1}, ..., \omega_{i+k-1} \in \pi_j$.

Assumption 1 means that each information set is an "interval" in that it contains only consecutive elements of Ω (there are no "holes").

Assumption 2: The partitions Π_1 and Π_2 are *overlapping, i.e.,* for any $\pi_1 \in \Pi_1$ and $\pi_2 \in \Pi_2$ the following holds:

$$\omega_1, \omega_n \notin \pi_1 \Longrightarrow \pi_1 \nsubseteq \pi_2$$

$$\omega_1, \omega_n \notin \pi_2 \Longrightarrow \pi_2 \nsubseteq \pi_1$$

Assumption 2 means that except in the extreme states of nature (containing the lowest and the highest values of winning), after observing their own signals neither player has an information advantage over his opponent.

Malueg and Orzach (2009) showed that the environment where the players partitions are connected and overlapping can be viewed in the following simpler but equivalent form:

Proposition 1 *Every common-value all-pay auction with information partitions* $\Pi_i, i = 1, 2$ *which are connected and overlapping is strategically equivalent to a common-value all-pay auction with information partitions given by*

$$\begin{aligned} \Pi_1 &= \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_5\}, ..., \{\omega_{n-1}, \omega_n\}\} \\ \Pi_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, ..., \{\omega_{n-2}, \omega_{n-1}\}, \{\omega_n\}\} \end{aligned} \tag{3}$$

In accordance with Proposition 1, we will assume from now on that information partitions Π_1 and Π_2 are given by (3), unless stated otherwise.

It can be shown that some environments not satisfying Assumptions 1 and 2 can nevertheless be reduced to strategically equivalent environments that do satisfy Assumptions 1 and 2. For this purpose consider the following example given by Malueg and Orzach (2009). In that example $\Omega = \{1, 2, 3, 4, 5\}, \Pi_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}, \Pi_2 = \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}$, and $v_i = i$ for every $i = 1, ..., 5$. It obviously does not satisfy Assumption 2. However, denote $\widetilde{\Omega} = \{1, 2, 3\}$ and $\widehat{\Omega} = \{4, 5\}$. It is clear that regardless of the realized state, it is common knowledge whether the state is in $\widetilde{\Omega}$ or $\widehat{\Omega}$. Note that each of $\widetilde{\Omega}$ and $\widehat{\Omega}$ together with the partitions induced by Π_1 and Π_2 satisfies Assumption 2 and therefore this environment is strategically

equivalent (conditional on a realization of either $\widetilde{\Omega}$ or $\widehat{\Omega}$) to the common-value all-pay auctions with information partitions given by (3). Malueg and Orzach (2009) provided other examples depicting environments that can be transformed into the setting analyzed in this paper. Furthermore, it is worth noting that our analysis remains valid if Ω is an *infinite* set of states of nature provided the partitions are finite. To see this, simply replace Ω with a finite Ω', which is the coarsest partition of Ω that refines all $\{\Pi_n\}_{n\in\mathcal{N}}$, and, for each $\pi \in \Omega'$, let the value of winning at π, $v(\pi)$, be equal to the conditional expectation $E(v(\cdot) \mid \pi)$. Hence, all-pay auctions with the information structure satisfying Assumptions 1 and 2 are suitable to analyze a larger family of all-pay auctions, with more general information partitions.

In order to describe now equilibrium strategies we introduce the following notations. Denote

$$\pi^i \equiv \begin{cases} \{\omega_i, \omega_{i+1}\}, & \text{if } i = 1, 2, ..., n-1, \\ \{\omega_1\}, & \text{if } i = 0, \\ \{\omega_n\}, & \text{if } i = n. \end{cases}$$

Thus, Π_1 consists of the sets π^i for every even integer $0 \leq i \leq n-1$, and Π_2 consists of the sets π^i for every odd integer $1 \leq i \leq n$. Also, for every $i = 1, ..., n-1$, let

$$p^{i,i+1} \equiv p(\omega_i \mid \pi^i) \ (= \frac{p_i}{p_i + p_{i+1}})$$

and

$$p^{i+1,i} \equiv 1 - p^{i,i+1}$$

be the conditional probabilities of the states ω_i and ω_{i+1} given the event π^i. Additionally, set

$$p^{n,n+1} = p^{1,0} \equiv 1, \ p^{n+1,n} = p^{0,1} \equiv 0.$$

In what follows, we describe a mixed strategy equilibrium (F_1^{**}, F_2^{**}) of the all-pay auction. Let $x_0 = 0$, and for every $i = 1, 2..., n$ set

$$x_i \equiv \sum_{j=1}^{i} p^{j,j-1} p^{j,j+1} v_j.$$

Given $\pi^0 = \{\omega_1\}$, player 1's mixed strategy is

$$F_1^{**}\left(\pi^0, x\right) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{p^{1,2}v_1}, & \text{if } 0 \leq x \leq x_1, \\ 1, & \text{if } x > x_1. \end{cases} \tag{4}$$

Note that the function $F_1^{**}\left(\pi^0, \cdot\right)$ is well defined, strictly increasing on $[x_0, x_1]$, continuous, $F_1^{**}\left(\pi^0, x_0\right) = 0$ and $F_1^{**}\left(\pi^0, x_1\right) = 1$. Thus, $F_1^{**}\left(\pi^0, \cdot\right)$ is a c.d.f. of a continuous probability distribution supported on $[x_0, x_1]$.

For $i = 1, 2, ..., n-1$, given $\pi^i = \{\omega_i, \omega_{i+1}\}$ and assuming that $\pi^i \in \Pi_j$ for player j, the mixed strategy of player j is

$$F_j^{**}\left(\pi^i, x\right) = \begin{cases} 0, & \text{if } x < x_{i-1}, \\ \frac{x - x_{i-1}}{p^{i,i-1}v_i}, & \text{if } x_{i-1} \leq x \leq x_i, \\ \frac{x + p^{i+1,i+2}v_{i+1} - x_{i+1}}{p^{i+1,i+2}v_{i+1}}, & \text{if } x_i < x \leq x_{i+1}, \\ 1, & \text{if } x > x_{i+1}. \end{cases} \tag{5}$$

The function $F_j^*\left(\pi^i, \cdot\right)$ is well defined, strictly increasing on $[x_{i-1}, x_{i+1}]$, continuous, $F_j^{**}\left(\pi^i, x_{i-1}\right) = 0$, $F_j^{**}\left(\pi^i, x_{i+1}\right) = 1$ and $F_j^{**}\left(\pi^i, x_i\right) = p^{i,i+1}$. Thus, $F_j^{**}\left(\pi^i, \cdot\right)$ is a c.d.f. of a continuous probability distribution supported on $[x_{i-1}, x_{i+1}]$.

Finally, given $\pi^n = \{\omega_n\}$, player 2's mixed strategy is

$$F_2^{**}(\pi^n, x) = \begin{cases} 0, & \text{if } x < x_{n-1}, \\ \frac{x - x_{n-1}}{p^{n,n-1}v_n}, & \text{if } x_{n-1} \leq x \leq x_n, \\ 1, & \text{if } x > x_n \ . \end{cases} \tag{6}$$

The function $F_2^{**}\left(\pi^n, \cdot\right)$ is well defined, strictly increasing on $[x_{n-1}, x_n]$, continuous, $F_2^{**}\left(\pi^n, x_{n-1}\right) = 0$ and $F_2^{**}\left(\pi^n, x_n\right) = 1$. Thus, $F_2^{**}\left(\pi^n, \cdot\right)$ is a c.d.f. of a probability distribution supported on $[x_{n-1}, x_n]$.

Proposition 2 *Suppose that*

$$p^{i+1,i+2}v_{i+1} - v_i \geq 0 \text{ for every } i = 1, 2, ..., n-1. \tag{7}$$

Then, the strategy profile $F^{**} = (F_1^{**}, F_2^{**})$ described in (4), (5), and (6) is a mixed strategy equilibrium of

the common-value all-pay auction G.[7]

Proof. See Appendix. ■

Condition (7) is satisfied whenever each value v_{i+1} is sufficiently above its closest predecessor v_i. It is easy to see that the pair F^{**} need not be an equilibrium if (7) does not hold (in particular, when $v_1 = v_2 = ... = v_n$).

5 Results

In this section we derive some comparative results about the players' expected payoffs and efforts in equilibrium F^{**} (described in (4), (5), and (6) in the previous section). The proof of Proposition 2 (given in the Appendix) provides an explicit formula for expected equilibrium payoffs of the players conditional on each event π^i, for $i = 0, ..., n$. If j is the player for whom $\pi^i \in \Pi_j$, it follows from (11), (12), and (13) that[8]

$$E_j(\pi^i, F^{**}) = \sum_{k=1}^{i-1} p^{k,k+1}(p^{k+1,k+2}v_{k+1} - v_k), \tag{8}$$

when the sum in (8) is defined as 0 if $i \leq 1$. This yields the following immediate result that compares the players' conditional expected payoffs.

Proposition 3 *Under condition (7),*

1) for every $i = 1, ..., n$, and player j such that $\pi^i \in \Pi_j$,[9]

$$E_j(\pi^i, F^{**}) - E_{-j}(\pi^{i-1}, F^{**}) = p^{i-1,i}(p^{i,i+1}v_i - v_{i-1}) \geq 0.$$

It follows, in particular, that

2) the expected payoff of each player j, conditional on $\pi^i \in \Pi_j$, is increasing in i:

$$E_1(\pi^0, F^{**}) \leq E_1(\pi^2, F^{**}) \leq ... \leq E_1(\pi^{n-1}, F^{**})$$

[7] Note that (7) can be assumed to hold only for $i = 1, 2, ..., n-2$. As for $i = n-1$, the inequality $p^{n,n+1}v_n - v_{n-1} = v_n - v_{n-1} \geq 0$ holds trivially by our assumption that $v_i \geq v_{i-1}, 1 < i \leq n$.

[8] We additionally use the fact that, by definition, $F_j^{**}(\pi^i, \cdot)$ is supported on $[x_{i-1}, x_{i+1}]$ if $1 \leq i \leq n-1$ and $\pi^i \in \Pi_j$, $F_1^{**}(\pi^0, \cdot)$ is supported on $[x_0, x_1]$, and $F_2^{**}(\pi^n, \cdot)$ is supported on $[x_{n-1}, x_n]$.

[9] As in the proof of Proposition 2, we use the convention that when $i = 1$, $v_{i-1} = v_0$ is defined as 0.

and

$$E_2(\pi^1, F^{**}) \leq E_2(\pi^3, F^{**}) \leq ... \leq E_2(\pi^n, F^{**}).$$

Furthermore,

3) no player is dominant with respect to the expected payoff, i.e., for every $i = 2, 4, ..., n-1$[10]

$$E_1(\pi_1(\omega_i), F^{**}) \geq E_2(\pi_2(\omega_i), F^{**})$$

and for every $i = 1, 3, ..., n$

$$E_1(\pi_1(\omega_i), F^{**}) \leq E_2(\pi_2(\omega_i), F^{**}).$$

The next result shows that although the players are asymmetrically informed (i.e., have different information partitions), their ex-ante distributions of equilibrium effort are identical.

Proposition 4 *In equilibrium* F^{**}*, both players have the same* ex-ante *distribution of effort. In particular, the expected efforts of both players are the same.*

Proof. Let $x_{i-1} \leq x \leq x_i$, for $i = 2, ..., n-1$, and let j be the player for whom $\pi^i \in \Pi_j$. Then the ex-ante probability that player j exerts an effort smaller than or equal to x is

$$\begin{aligned} \overline{F}_j^{**}(x) &= \sum_{k=1}^{i-1} p_k + (p_i + p_{i+1}) F_j^{**}\left(\pi^i, x\right) \\ &= \sum_{k=1}^{i-1} p_k + (p_i + p_{i+1}) \frac{x - x_{i-1}}{p^{i,i-1} v_i} \\ &= \sum_{k=1}^{i-1} p_k + \frac{(p_i + p_{i+1})(p_i + p_{i-1})(x - x_{i-1})}{p_i v_i}. \end{aligned} \tag{9}$$

The ex-ante probability that j's rival, player $-j$, exerts an effort smaller than or equal to x is then

$$\begin{aligned} \overline{F}_{-j}^{**}(x) &= \sum_{k=1}^{i-2} p_k + (p_{i-1} + p_i) F_{-j}^{**}\left(\pi^{i-1}, x\right) \\ &= \sum_{k=1}^{i-2} p_k + (p_{i-1} + p_i) \frac{x + p^{i,i+1} v_i - x_i}{p^{i,i+1} v_i} \\ &= \sum_{k=1}^{i-1} p_k + \frac{(p_i + p_{i+1})(p_i + p_{i-1})(x - x_{i-1})}{p_i v_i} \\ &= \overline{F}_j^{**}(x). \end{aligned} \tag{10}$$

[10] Recall that $\pi_j(\omega_i) \in \Pi_j$ denotes the element of Π_j that contains ω_i.

With the convention that $\sum_{k=1}^{i-2} p_k = \sum_{k=1}^{i-1} p_k = 0$ when $i < 2$ and that $p_0 = p_{n+1} = 0$, (9) and (10) also hold for $i = 1$ and $i = n$.

Since we showed that $\overline{F}_1^{**}(x) = \overline{F}_2^{**}(x)$ for every $x \in [0, x_n]$, and since obviously $\overline{F}_1^{**}(x) = \overline{F}_2^{**}(x) = 1$ for $x > x_n$, the ex-ante distributions of effort in equilibrium F^{**} are identical for both players. ■

It is worth noting that although the players have the same ex-ante distribution of effort, we showed that their ex-ante probabilities to win the contest are not the same.

6 Concluding remarks

In models with asymmetric information, differences in players' information usually result in different equilibrium strategies, probabilities of winning, and expected payoffs. In our model where no player has information advantage over his opponent, we show that equilibrium efforts have the same distribution for all players, and thus the expected efforts of all players are equal. We also show that the conditional expected payoffs of the players increase in the expected value of winning (as conveyed by the revealed information).

We present sufficient conditions for the existence of the equilibrium in our model. One of them is that the information set of each player is connected with respect to the value of winning the contest. This condition is also sufficient for the existence of equilibrium in the framework when the different information endowments can be ranked (see Einy et al (2013a)). Our results yield the conclusion that in common-value all-pay auctions the players' information does not affect the ratio of the players' expected efforts. It is interesting to note that this property does not necessarily hold for other forms of contests such as Tullock contests with or without the assumption of information advantage (see Einy et al. (2013b) and Warneryd (2003, 2012)).

7 Appendix

Proof of Proposition 2

For any $i = 0, ..., n$ consider the player j for whom $\pi^i \in \Pi_j$, and assume that j's rival (denoted $-j$) uses the strategy F_{-j}^{**}. The expected payoff of player j conditional on the event π^i is given as follows. If $1 \leq$

$i \leq n$, and j exerts effort $x \in [x_{i-1}, x_i]$, then

$$\begin{aligned} E_j(\pi^i, x, F_{-j}^{**}) &= \\ &= p^{i,i+1} v_i F_{-j}^{**}\left(\pi^{i-1}, x\right) - x \\ &= p^{i,i+1} v_i \frac{x + p^{i,i+1} v_i - x_i}{p^{i,i+1} v_i} - x \\ &= p^{i,i+1} v_i - x_i. \end{aligned} \tag{11}$$

If $1 \leq i \leq n-1$ and j exerts effort $x \in [x_i, x_{i+1}]$, then

$$\begin{aligned} E_j(\pi^i, x, F_{-j}^{**}) &= \\ &= p^{i,i+1} v_i + p^{i+1,i} v_{i+1} F_{-j}^{**}\left(\pi^{i+1}, x\right) - x \\ &= p^{i,i+1} v_i + p^{i+1,i} v_{i+1} \frac{x - x_i}{p^{i+1,i} v_{i+1}} - x \\ &= p^{i,i+1} v_i - x_i. \end{aligned} \tag{12}$$

Now set $v_0 = 0$. Then (12) applies also when $i = 0$ (in which case $j = 1$), i.e. (12) holds for every $0 \leq i \leq n-1$.

Equalities (11) and (12) establish the following fact:

Fact 1. When player j's opponent uses F_{-j}^{**}, player j is: (i) indifferent between all efforts in $[x_{i-1}, x_{i+1}]$ given the event $\pi^i \in \Pi_j$ for $1 \leq i \leq n-1$; (ii) indifferent between all efforts in $[x_0, x_1]$ given π^0 (if j is player 1); (iii) indifferent between all efforts in $[x_{n-1}, x_n]$ given π^n (if j is player 2).

It can be shown by induction on i that, for $i = 2, 3, ..., n$,

$$p^{i,i+1} v_i - x_i = \sum_{k=1}^{i-1} p^{k,k+1}(p^{k+1,k+2} v_{k+1} - v_k) \geq 0. \tag{13}$$

The expression in (13) is non-negative as every summand in $\sum_{k=1}^{i-1} p^{k,k+1}(p^{k+1,k+2} v_{k+1} - v_k)$ is non-negative by assumption (7). When $i = 0$ or $i = 1$, equality (13) remains meaningful if the sum is defined as 0. It then follows from (13) and (11), (12) that:

Fact 2. The conditional expected payoffs of player j considered in (11) and (12) are non-negative for the corresponding efforts.

Next consider $\pi^i \in \Pi_j$, for some $0 \leq i \leq n-2$ and player j. Notice that, given the event π^i, if

$y \in [x_{i+1}, x_{i+2}]$ then

$$\begin{aligned} E_j(\pi^i, y, F_{-j}^{**}) &= p^{i,i+1}v_i + p^{i+1,i}v_{i+1}F_{-j}^{**}\left(\pi^{i+1}, y\right) - y \\ &= p^{i,i+1}v_i + p^{i+1,i}v_{i+1}\frac{y + p^{i+2,i+3}v_{i+2} - x_{i+2}}{p^{i+2,i+3}v_{i+2}} - y \\ &\leq p^{i,i+1}v_i + p^{i+1,i}v_{i+1}\frac{x_{i+1} + p^{i+2,i+3}v_{i+2} - x_{i+2}}{p^{i+2,i+3}v_{i+2}} - x_{i+1} \\ &= p^{i,i+1}v_i + p^{i+1,i}v_{i+1}\frac{x_{i+1} - x_i}{p^{i+1,i}v_{i+1}} - x_{i+1} \\ &= p^{i,i+1}v_i - x_i = E_j(\pi^i, x_i, F_{-j}^{**}). \end{aligned} \tag{14}$$

The inequality in (14) holds since, by (7), $p^{i+2,i+3}v_{i+2} > p^{i+1,i}v_{i+1}$, and the last equality in (14) holds by (12). Since obviously, if $y > x_{i+2}$ and $i \leq n-2$,

$$E_j(\pi^i, y, F_{-j}^{**}) \leq E_j(\pi^i, x_{i+2}, F_{-j}^{**}) \tag{15}$$

and, if $y > x_n$,

$$\begin{aligned} E_1(\pi^{n-1}, y, F_2^{**}) &\leq E_1(\pi^{n-1}, x_n, F_2^{**}) = E_1(\pi^{n-1}, x_{n-1}, F_2^{**}), \\ E_2(\pi^n, y, F_1^{**}) &\leq E_2(\pi^n, x_n, F_1^{**}) \end{aligned} \tag{16}$$

Then (15), (16) and (14) establish the following:

Fact 3. When player j's rival uses F_{-j}^{**}, player j (weakly) prefers effort x_i to any effort above $x_{\min(i+1,n)}$, given the event $\pi^i \in \Pi_j$ for $0 \leq i \leq n$.

Now consider $\pi^i \in \Pi_j$, for some $2 \leq i \leq n$ and player j. Given the event π^i, if $y \in [x_{i-2}, x_{i-1}]$ then

$$\begin{aligned} E_j(\pi^i, y, F_{-j}^{**}) &= \\ &= p^{i,i+1}v_i F_{-j}^{**}\left(\pi^{i-1}, y\right) - y \\ &= p^{i,i+1}v_i\frac{y - x_{i-2}}{p^{i-1,i-2}v_{i-1}} - y \\ &\leq p^{i,i+1}v_i\frac{x_{i-1} - x_{i-2}}{p^{i-1,i-2}v_{i-1}} - x_{i-1} \\ &= p^{i,i+1}v_i - x_i = E_j(\pi^i, x_i, F_{-j}^{**}). \end{aligned} \tag{17}$$

The inequality in (17) holds since by (7) $p^{i,i+1}v_i \geq p^{i-1,i-2}v_{i-1}$, and the last equality in (17) holds by (11). Note also that when $i \geq 2$ and $0 \leq y \leq x_{i-2}$,

$$E_j(\pi^i, y, F_{-j}^{**}) \leq 0. \tag{18}$$

Then (17), (18), and Fact 2 lead to the following:

Fact 4. When player j's rival uses F_{-j}^{**}, player j (weakly) prefers effort x_i to any effort below x_{i-1}, given the event $\pi^i \in \Pi_j$ for $2 \leq i \leq n$.

Facts 1, 3, and 4 show that for any $0 \leq i \leq n$, conditional on the event $\pi^i \in \Pi_j$, any effort in the support of $F_j^{**}\left(\pi^i, \cdot\right)$ is a best response of player j against the mixed strategy F_{-j}^{**} of his rival. Thus F_j^{**} is also an unconditional best response of player j, which means that F^{**} is indeed an equilibrium of G. Q.E.D.

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