Extensions of Flux Theory

Reuven Segev

Department of Mechanical Engineering
Ben-Gurion University

Department of Mechanical and Aerospace Engineering
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Objects of Interest

- *Fluxes and stresses* as fundamental objects of continuum mechanics.
- *Geometric aspects:* Formulations that do not use the traditional geometric and kinematic assumptions. For example, Euclidean structure of the physical space, mass conservation. Materials with micro-structure (sub-structure), growing bodies.
- *Analytic aspects:* Irregular bodies and flux fields. Fractal bodies.

Flux Theory?

Derive the existence of the flux vector field $\boldsymbol{j}$, e.g., the heat flux vector field or the electric current density, and its properties from global balance laws, e.g., balance of energy or conservation of charge.

Relevant Operations:

- *Total Flux (Flow) Calculation:*

$$\int_A \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{d}A.$$

- *Gauss-Green Theorem:*

$$\int_{\partial B} \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{d}A = \int_B \operatorname{div} \boldsymbol{j} \, \mathrm{d}V.$$

Questions Regarding the Operations

- *Total Flux Calculation:*

$$\int_A \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{d}A.$$

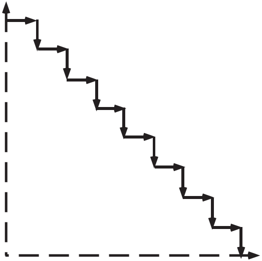
  - How irregular can A be?

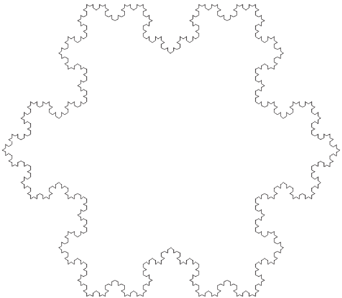
- *Gauss-Green Theorem:*

$$\int_{\partial B} \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{d}A = \int_B \operatorname{div} \boldsymbol{j} \, \mathrm{d}V.$$

  - How irregular can B be?
  - How irregular can $\boldsymbol{j}$ be?

Examples:





Balanced Extensive Properties

In terms of scalar extensive property p with density ρ in space, one assumes for every "control region" $B \subset \mathscr{U} \cong \mathbb{R}^3$:

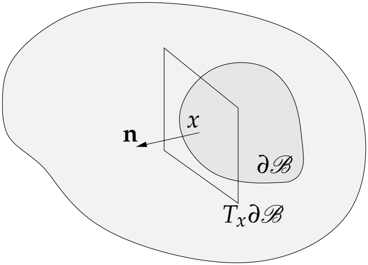
- Consider β, interpreted as the *time derivative* of the density ρ of the property, so for any control region B in space, $\int_B \beta \mathrm{d}V$ is the rate of change of the total content of the property inside B.
- For each control region B there is a *flux density* τ_B such that $\int_{\partial B} \tau_B \mathrm{d}A$ is the *total flux (flow)* of the property out of B.
- There is a function s on $\mathscr{U}$ such that for each region B

$$\int_B \beta \, \mathrm{d}V + \int_{\partial B} \tau_B \, \mathrm{d}A = \int_B s \, \mathrm{d}V.$$

Here, s is interpreted as the *source density* of the property p (e.g., $s = 0$ for mass and electric charge).

Fluxes: Traditional Cauchy Postulate and Theorem

Cauchy's postulate and theorem are concerned with the dependence of τ_B on B.



- It uses the metric properties of space.
- $\tau_B(x)$ is assumed to depend on B only through the unit normal to the boundary at x.
- The resulting Cauchy theorem asserts the existence of the flux vector $\boldsymbol{j}$ such that $\tau_B(x) = \boldsymbol{j} \cdot \boldsymbol{n}$.

Assumptions Again:

In terms of a scalar extensive property with density ρ in space, one assumes that there are operators $T(\partial B)$, the *total flux operator*, and $S(B)$ the *total content* operator, such that for every "control region" $B \subset \mathscr{U} \cong \mathbb{R}^3$ (we take $s = 0$):

- *Balance*: $T(\partial B) + S(B) = 0$
- *Regularity*: $S(B) = \int_B \beta_B \, \mathrm{d}V$, and $T(\partial B) = \int_{\partial B} \tau_B \, \mathrm{d}A$
- *Locality (pointwise)*: $\beta_B(x) = \beta(x)$, and $\tau_B(x) = \tau(x, \boldsymbol{n})$
- *Continuity*: $\tau(\cdot, \boldsymbol{n})$ is continuous.

Note: It follows from the balance and regularity assumptions that

- $|\partial B| \to 0$ implies $T(\partial B) \to 0$,
- $|B| \to 0$ implies $T(\partial B) \to 0$

$|\cdot|$ being either the area or volume depending on the context.

The Results:

Cauchy's Theorem

asserts that $\tau(x, \boldsymbol{n})$ depends linearly on $\boldsymbol{n}$. There is a vector field $\boldsymbol{j}$ such that

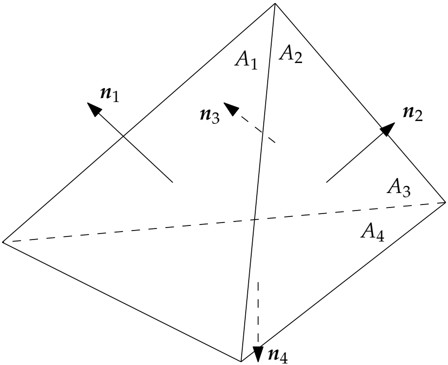
$$\tau = \boldsymbol{j} \cdot \boldsymbol{n}.$$

Considering smooth regions and flux vector fields such that Gauss-Green theorem may be applied, the balance may be written in the form of a differential equation as

$$\operatorname{div} \boldsymbol{j} + \beta = s.$$

Traditional Proof:

- Consider the infinitesimal tetrahedron. Since the area is in an order of magnitude larger than the volume, the volume terms are negligible.
- Thus, $\sum_i A_i \tau(\boldsymbol{n}_i) = 0$.
- Also, $\sum_i A_i \boldsymbol{n}_i = 0$.
- Hence,



$$\tau\left(\frac{A_1}{A_4}\boldsymbol{n}_1 + \frac{A_2}{A_4}\boldsymbol{n}_2 + \frac{A_3}{A_4}\boldsymbol{n}_3\right) = \frac{A_1}{A_4}\tau(\boldsymbol{n}_1) + \frac{A_2}{A_4}\tau(\boldsymbol{n}_2) + \frac{A_3}{A_4}\tau(\boldsymbol{n}_3)$$

Contributions in Continuum Mechanics

- Noll: 1957, 1973, 1986,
- Gurtin & Williams: 1967,
- Gurin & Martins: 1975,
- Gurtin, Williams & Ziemer: 1986,
- Silhavy: 1985, 1991, . . ., 2007,
- Noll & Virga: 1988,
- Degiovanni, Marzocchi & Musesti: 1999, . . .
- Fosdick & Virga: 1989.
- Segev: 1986, 1991, 1999, 2000, 2002.

The Proposed Formulation

Uses *Geometric Integration Theory* by Whitney (1957).

- Building blocks: r-dimensional oriented cells in E^n.
- Formal vector space of r-cells: polyhedral r-chains.
- Complete w.r.t a norm: Banach space of r-chains.
- Elements of the dual space: r-cochains.

Relevance to Flux Theory

- The total flux operator on regions is modelled mathematically by a cochain.
- Cauchy's flux theorem is implied by a representation theorem for cochains by forms.

Features of the Proposed Formulation

- It offers a common point of view for the analysis of the following aspects: *class of domains, integration, Stokes' Theorem, and fluxes.*
- Allows irregular domains and flux fields.
- The co-dimension not limited to 1. Allows membranes, strings, etc. Not only the boundary is irregular, but so is the domain itself.
- Compatible with the formulation on general manifolds where no particular metric is given.

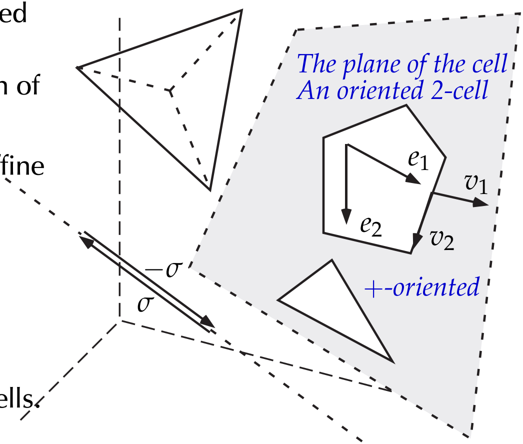
Outline

- Cells and polyhedral chains
- Algebraic cochains
- Norms and the complete spaces of chains
- The representation of cochains by forms:
  - Multivectors and forms
  - Integration
  - Representation
  - Coboundaries and differentiable balance equations

Cells and Polyhedral Chains

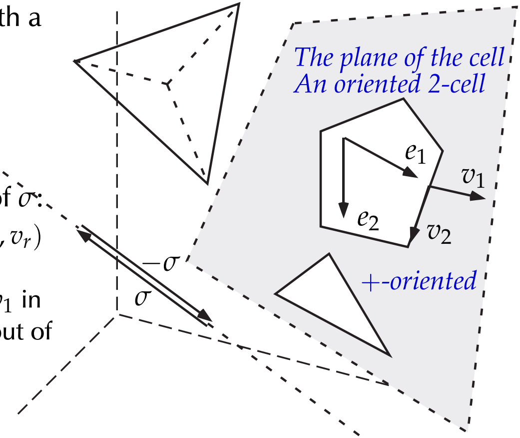
Oriented Cells

- A *cell*, σ, is a non empty bounded subset of E^n expressed as an intersection of a finite collection of half spaces.
- The *plane of* σ is the smallest affine subspace containing σ.
- The *dimension* r of σ is the dimension of its plane. Terminology: an r-cell.
- The boundary $\partial\sigma$ of an r-cell σ contains a number of $(r-1)$-cells.



Oriented Cells (continued)

- Recall: An orientation of a vector space is determined by a choice of a basis. Any other basis will give the *same orientation* if the determinant of the transformation is positive. A vector space can have 2 orientations.
- An *oriented* r-cell is an r-cell with a choice of one of the two orientations of the vector space associated with its plane.
- The orientation of $\sigma' \in \partial\sigma$ is determined by the orientation of σ:
  - Choose independent $(v_2, \ldots, v_r)$ in σ'.
  - Order them such that given v_1 in the plane of σ which points out of σ', $(v_1, \ldots, v_r)$ are positively oriented relative to σ.



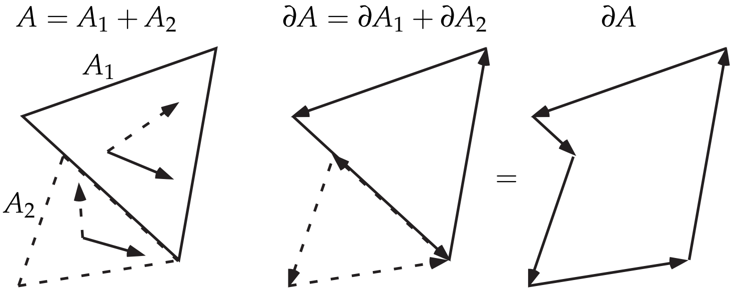
Polyhedral Chains: Algebra into Geometry

- A *polyhedral r-chain* in E^n is a formal linear combination of r-cells

$$A = \sum a_i \sigma_i.$$

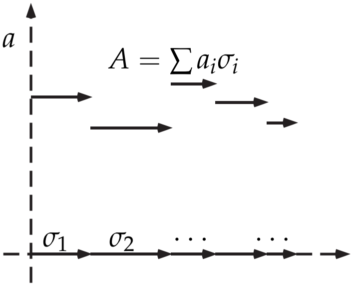
- The following operations are defined for polyhedral chains:
  - The polyhedral chain 1σ is identified with the cell σ.
  - We associate multiplication of a cell by -1 with the operation of inversion of orientation, i.e., $-1\sigma = -\sigma$.
  - If σ is cut into $\sigma_1, \ldots, \sigma_m$, then σ and $\sigma_1 + \ldots + \sigma_m$ are identified.
  - Addition and multiplication by numbers in a natural way.
- The space of polyhedral r-chains in E^n is now an *infinite-dimensional vector space* denoted by $\mathscr{A}_r(E^n)$.
- The *boundary of a polyhedral r-chain* $A = \sum a_i \sigma_i$ is $\partial A = \sum a_i \partial \sigma_i$. Note that ∂ is a linear operator $\mathscr{A}_r(E^n) \longrightarrow \mathscr{A}_{r-1}(E^n)$.

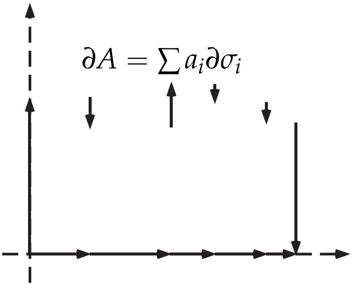
Polyhedral Chains: Illustration



$$\partial \colon \mathscr{A}_r \to \mathscr{A}_{r-1}$$

A Polyhedral Chain as a Function





Total Fluxes as Cochains

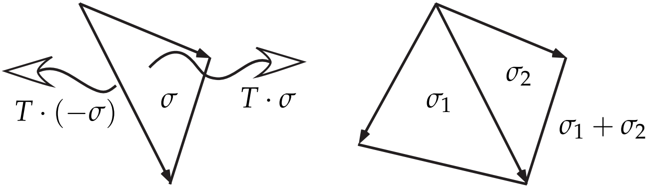
Basic Idea:

Regard the flux through a 2-dimensional chain as the action of a linear operator—a *co-chain*—on that chain.

A *cochain:* Linear $T\colon \mathscr{A}_r \to \mathbb{R}$. We write $T(B) = T \cdot B$.

Algebraic implications:

- additivity,
- interaction antisymmetry.



$$T \cdot (-\sigma) = -T \cdot \sigma, \qquad T \cdot (\sigma_1 + \sigma_2) = T \cdot \sigma_1 + T \cdot \sigma_2$$

Norms and the Complete Space of Chains: Analysis into Geometry

The Norm Induced by Boundedness

Boundedness: $|T_{\partial B}| \leqslant N_2 |\partial B|$, $|T_{\partial B}| \leqslant N_1 |B|$. Setting $A = \partial B$, ...

As a cochain: $|T \cdot A| \leqslant N_2 |A|$, $|T \cdot \partial D| \leqslant N_1 |D|$, $A \in \mathscr{A}_r$, $D \in \mathscr{A}_{r+1}$.

Thus, for any $D \in \mathscr{A}_{r+1}$, and $A \in \mathscr{A}_r$:

$$\begin{aligned} |T \cdot A| &= |T \cdot A - T \cdot \partial D + T \cdot \partial D| \\ &\leqslant |T \cdot A - T \cdot \partial D| + |T \cdot \partial D| \\ &\leqslant N_2 |A - \partial D| + N_1 |D| \\ &\leqslant C_T (|A - \partial D| + |D|), \end{aligned}$$

Basic Idea (revised)

Regard the flux as a *continuous linear functional* on the space of chains w.r.t. a norm

$$|T \cdot A| \leqslant C_T \|A\|,$$

where the *flat norm* (smallest) is given as

$$\|A\| = |A|^{\flat} = \inf_D \{|A - \partial D| + |D|\}.$$

Flat Chains

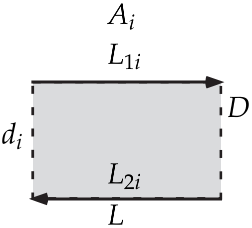
- The *mass* of a polyhedral r-chain $A = \sum a_i \sigma_i$ is $|A| = \sum |a_i||\sigma_i|$.
- The *flat norm*, $|A|^\flat$, of a polyhedral r-chain:

$$|A|^\flat = \inf\{|A - \partial D| + |D|\},$$

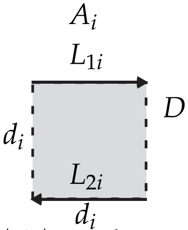
using all polyhedral $(r+1)$-chains D.

-
  - Taking $D = 0$, $|A|^\flat \leqslant |A|$.
  - If $A = \partial B$, taking $D = B$ gives $|A|^\flat \leqslant |B|$. Hence, $|\partial B|^\flat \leqslant |B|$.
- Completing $\mathscr{A}_r(E^n)$ w.r.t. the flat norm gives a Banach space denoted by $\mathscr{A}_r^\flat(E^n)$, whose elements are *flat* r-chains in E^n.
- Flat chains may be used to represent continuous and smooth submanifolds of E^n and even irregular surfaces.
- The *boundary of a flat $(r+1)$-chain* $A = \lim^\flat A_i$, is the a flat r-chain $\partial A = \lim \partial A_i$. The boundary operator is continuous and linear.

Flat Chains, an Example (Illustration - I):

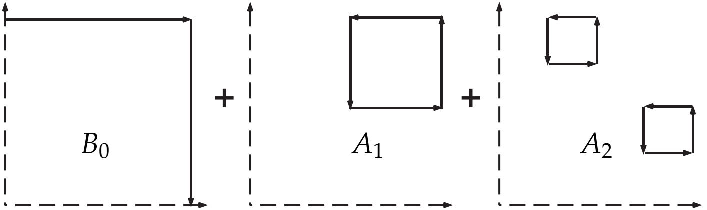


$$|A_i| = 2L,$$
$$|A_i|^\flat \leqslant (L+2)d_i \to 0.$$

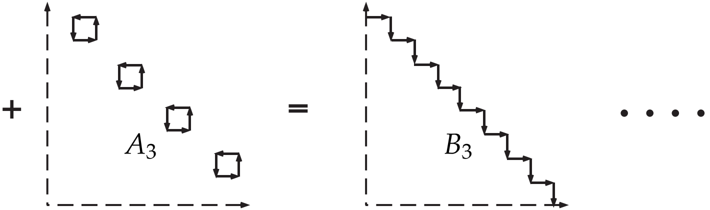


$$|A_i| = 2d_i,$$
$$|A_i|^\flat \leqslant 2d_i + d_i^2 \to 0.$$

Example: The Staircase



The dashed lines are for reference only.

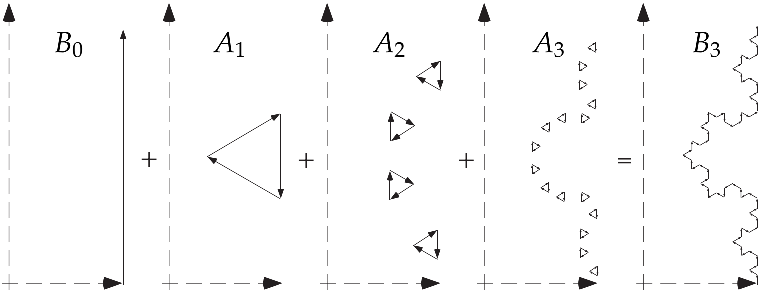


$$|A_i|^\flat \leqslant 2^{i-1}2^{-2i} = 2^{-i}/2 \quad \Longrightarrow \quad (B_i) \text{ a convergent series.}$$

Note, $\left|B_i - B_j\right| = \left|\sum_{k=j+1}^{i} A_k\right| \leq \sum_{k=j+1}^{i} |A_k| \leq \sum_{k=j+1}^{\infty} |A_k| \leqslant \sum_{k=j+1}^{\infty} 2^{-k}/2, \quad \forall \quad i > j.$

Example: the Van Koch Snowflake

A_i contains 4^i triangles of side length 3^{-i}. Each time the length increases by $2 \cdot 3^{-i} \cdot 4^i = 2\left(\frac{4}{3}\right)^i$. Hence, $|B_i| \to \infty$.



$$|A_i|^\flat \leqslant 4^i \frac{\sqrt{3}}{2} 3^{-i} 3^{-i} = \frac{\sqrt{3}}{2}\left(\frac{2}{3}\right)^i$$

The Representation of Cochains by Forms

Objectives:

- Create an algebraic language to treat chains and cochains,
- A representation theorem for cochains in terms of fields and integration.

Multivectors

- A *simple r-vector* in V is an expression of the form $v_1 \wedge \cdots \wedge v_r$, where $v_i \in V$.
- An r-*vector* in V is a formal linear combination of simple r-vectors, together with:

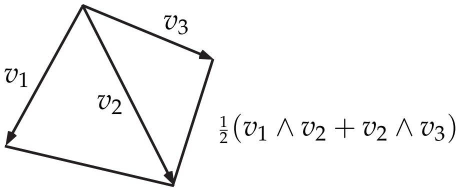
$$\begin{aligned}
(1)\quad & v_1 \wedge \cdots \wedge (v_i + v_i') \wedge \cdots \wedge v_r \\
& \qquad = v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r + v_1 \wedge \cdots \wedge v_i' \wedge \cdots \wedge v_r; \\
(2)\quad & v_1 \wedge \cdots \wedge (av_i) \wedge \cdots \wedge v_r = a(v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r); \\
(3)\quad & v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_r \\
& \qquad = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_r.
\end{aligned}$$

- The r-vector vanishes if the vectors are linearly dependent.
- The collection, V_r, of r-vectors is a vector space and $\dim V_r = \frac{n!}{(n-r)!r!}$.
- Given a basis $\{e_i\}$ of V, the r-vectors $\{e_{\lambda_1 \dots \lambda_r} = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_r}\}$, such that $1 \leq \lambda_1 < \cdots < \lambda_r \leq n$, *form a basis* of V_r.

The Representation of Polyhedral Chains by Multivectors

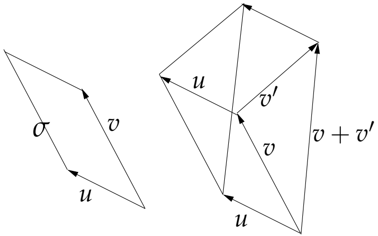
- Given an oriented r-*simplex* σ in E^n, with vertices $\{p_0 \dots p_r\}$, the r-*vector of* σ, $\{\sigma\}$, is $\{\sigma\} = v_1 \wedge \cdots \wedge v_r / r!$, where the v_i are defined by $v_i = p_i - p_0$ and are ordered such that they belong to σ's orientation.
  $\{\sigma\}$ represents the *plane*, *orientation* and *size* of σ—the relevant aspects.
- The *r-vector of a polyhedral r-chain* $\sum a_i \sigma_i$, is

$$\{\sum a_i \sigma_i\} = \sum a_i \{\sigma_i\}.$$



Why an r-covector?

For the 3-dimensional example, we want to measure the flux through any infinitesimal cell σ, $\{\sigma\} = v \wedge u$.



- Denote by $T(\sigma)$ the flux through that infinitesimal element.
- As $T(\sigma)$ depends only the plane, orientation and area, we expect

$$T(\sigma) = \widehat{T}(\{\sigma\}).$$

- Balance: $\widehat{T}$ is linear

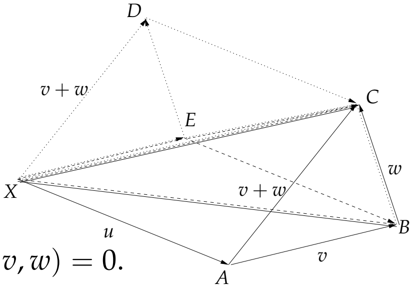
$$\widehat{T}(\sigma) = \tau \cdot \{\sigma\},$$

where τ is a linear mapping of multi-vectors to real numbers—an r-*covector*.

Rough Proof

Consider the infinitesimal tetrahedron X, A, B, C generated by the three vectors u, v, w.

— Use right-handed orientation.



— Balance implies:

$$T(v,u)+T(v,w)+T(u,v+w)-T(u+v,w)=0.$$

— Same for X, B, C, E and X, C, D, E

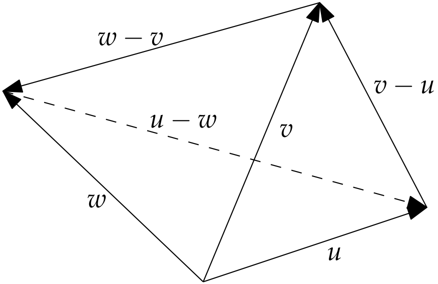
$$T(w,u)+T(u+v,w)+T(v,u)-T(v,w+u)=0$$
$$T(w,u)-T(v+w,u)-T(v,w)+T(v,w+u)=0.$$

— Add up to obtain: $T(u,v+w)=T(u,v)+T(u,w).$

Or Using Multi-Vectors

- Consider the infinitesimal tetrahedron D generated by the three vectors u, v, w and let $A = \partial D$.
- $|A|^\flat \leqslant |A - \partial D| + |D| \to 0$, as the volume of the tetrahedron decreases.
- Thus, $\lim T(\{A\}) = 0$.

— Use right-handed orientation.



Thus: $T(u \wedge v) + T(v \wedge w) + T(w \wedge u) + T((w - v) \wedge (v - u)) = 0.$

Using: $(w - v) \wedge (v - u) = w \wedge v - w \wedge u + v \wedge u = -u \wedge v - v \wedge w - w \wedge u,$

we conclude: $$\boxed{T(u \wedge v + v \wedge w + w \wedge u) = T(u \wedge v) + T(v \wedge w) + T(w \wedge u).}$$

Reminder: Dual Spaces of Vector Spaces

- For a vector space $\mathscr{W}$, $\mathscr{W}^*$—the *dual space*—is the collection of all linear mappings, $T : \mathscr{W} \longrightarrow \mathbb{R}$ (also *linear functionals, covectors*).
- In our case, flat chains are in $\mathscr{A}_r^\flat(E^n)$, and the total fluxes, being continuous linear functionals of chains, are $T \in \mathscr{A}_r^\flat(E^n)^*$.
- For an infinite dimensional vector space on which a norm $\|w\|$ is defined, one also requires that T is continuous. The condition for continuity (assuming linearity) is

$$|T(w)| \leqslant C_T \|w\|.$$

- This provides a procedure for generating new mathematical objects. Define a vector space and a norm and consider its dual space.
- *Representation Theorems*: represent the action of the linear functionals on vectors by known mathematical operations (inner products, integration).

Multi-Covectors

- An r-*covector* is an element of V^r—the dual space of V_r.
- r-covectors can be expressed using *covectors*:

$$V^r = (V^*)_r$$

$(V^*)_r$ is the space of *multi-covectors*, i.e., constructed as V_r using elements of the dual space V^*:

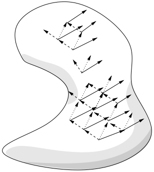
$$\tau = f_{\lambda_1 \cdots \lambda_r} e^{\lambda_1} \wedge \cdots \wedge e^{\lambda_r}, \quad \lambda_i < \lambda_{i+1}.$$

- r-covectors may be identified with *alternating multilinear* mappings:

$$V^r = L^r_A(V, \mathbb{R}), \quad \text{by} \quad \tau(v_1 \wedge v_2 \wedge \cdots \wedge v_r) = \bar{\tau}(v_1, \ldots, v_r).$$

- This is a simple example of a representation theorem for functionals.

Riemann Integration of Forms Over Polyhedral Chains



- An r-*form* in $Q \subset E^n$ is an r-covector valued mapping in Q.
- An r-form is continuous if its components are continuous functions.
- The *Riemann integral* of a continuous r-form τ over an r-*simplex* σ is defined as
$$\int_\sigma \tau = \lim_{k\to\infty} \sum_{\sigma_i \in \mathcal{S}_k\sigma} \tau(p_i) \cdot \{\sigma_i\},$$
where $\mathcal{S}_i\sigma$ is a sequence of *simplicial subdivisions* of σ with mesh $\to 0$, and each p_i is a point in σ_i.
- The Riemann integral of a continuous r-form over a *polyhedral r-chain* $A = \sum a_i\sigma_i$, is defined by $\int_A \tau = \sum a_i \int_{\sigma_i} \tau$.

Lebesgue Integral of Forms over Polyhedral Chains

- An r-form in E^n is *bounded and measurable* if all its components are bounded and measurable.
- The *Lebesgue integral* of an r-form τ over an r-cell σ is defined by

$$\int_\sigma \tau = \int_\sigma \tau(p) \cdot \frac{\{\sigma\}}{|\sigma|}\, dp,$$

where the integral on the right is a Lebesgue integral of a real function.
- This is extended by linearity to domains that are polyhedral chains by

$$\int_A \tau = \sum a_i \int_{\sigma_i} \tau,$$

for $A = \sum_i a_i \sigma_i$.

The Cauchy Mapping

- The *Cauchy mapping*, D_T, for the *cochain* T, gives $D_T(p, \alpha)$, at the *point* p *in the direction* α *defined by the cell* σ, defined as:

$$D_T(p, \alpha) = \lim_{i \to \infty} T \cdot \frac{\sigma_i}{|\sigma_i|}, \quad \alpha = \frac{\sigma_i}{|\sigma_i|}$$

  where all σ_i contain p, have r-direction α and $\lim_{i \to \infty} \operatorname{diam}(\sigma_i) = 0$.
- The Cauchy mapping for a given cochain T, of r-directions is analogous to the dependence of the flux density on the unit normal.

The Representation Theorem

Whitney:

- *The analog to Cauchy's flux theorem.* For each flat r-cochain T there is an r-form $\tau = \tau_T$ that represents T by

$$T \cdot A = \int_A \tau_T,$$

for every flat r-chain A.

Coboundaries and Balance Equations

- The *coboundary* dT of an r-cochain T is the $(r+1)$-cochain defined by

$$dT \cdot A = T \cdot \partial A.$$

  A very general form of "Stokes' theorem".

- Thus, d is the *dual of the boundary operator*:

$$\begin{array}{ccc} \mathscr{A}^\flat_{r+1}(E^n) & \xrightarrow{\partial} & \mathscr{A}^\flat_r(E^n) \\ \mathscr{A}^\flat_{r+1}(E^n)^* & \xleftarrow{d=\partial^*} & \mathscr{A}^\flat_r(E^n)^*. \end{array}$$

- The coboundaries of flat cochains are flat, as the boundary operator is continuous.
- Hence, there is a flat cochain S satisfying the global balance equation:

$$S \cdot A + T \cdot \partial A = 0, \quad \forall A, \quad \Longrightarrow \quad dT + S = 0.$$

  A very general form of the balance equation.

The Local Balance Equation

- If τ_T is a form that represents the total flux operator T, then, by the representation theorem applied to dT, there is a form representing dT

$$d_0\tau = \tau_{dT}.$$

- Thus,

$$dT \cdot B = T \cdot \partial B \quad \text{is represented by} \quad \int_B d_0\tau = \int_{\partial B} \tau_T.$$

- Let β be the r-form representing the rate of content operator S so

$$T(\partial B) + S(B) = 0 \quad \text{is represented by} \quad \int_{\partial B} \tau_T + \int_B \beta = 0.$$

- One obtains the local expression

$$d_0\tau + \beta = 0.$$

Stokes' Theorem for Differentiable Forms on Polyhedral Chains

- The *exterior derivative* of a *differentiable* r-form τ is an $(r+1)$-form $d\tau$ defined by

$$d\tau(p)\cdot(v_1\wedge\cdots\wedge v_{r+1})=\sum_{i=1}^{r+1}(-1)^{i-1}\nabla_{v_i}\tau(p)\cdot(v_1\wedge\cdots\wedge\widehat{v}_i\wedge\cdots\wedge v_{r+1}),$$

  where $\widehat{v}_i$ denotes a vector that has been omitted, and ∇_{v_i} is a directional derivative operator.

- Stokes' theorem for polyhedral chains, based on the fundamental theorem of differential calculus, states that

$$\int_A d\tau=\int_{\partial A}\tau$$

  for every differentiable r-form τ and an $(r+1)$-polyhedral chain A.

The Local Balance Equation for Differentiable Cochains

- Reminder:
  - If τ_T is a form that represents the total flux operator T, then, by the representation theorem applied to dT, there is a form representing dT

$$d_0\tau = \tau_{dT}.$$

  - One obtains the local expression

$$d_0\tau + \beta = 0.$$

- If τ_T is differentiable, then, $d_0\tau = d\tau$, the exterior derivative.

Thanks