Entanglement, thermodynamics & area

Ram Brustein

אוניברסיטת בן-גוריון

Series of papers with
Amos Yarom, BGU
(also David Oaknin, UBC)
hep-th/0302186 + to appear

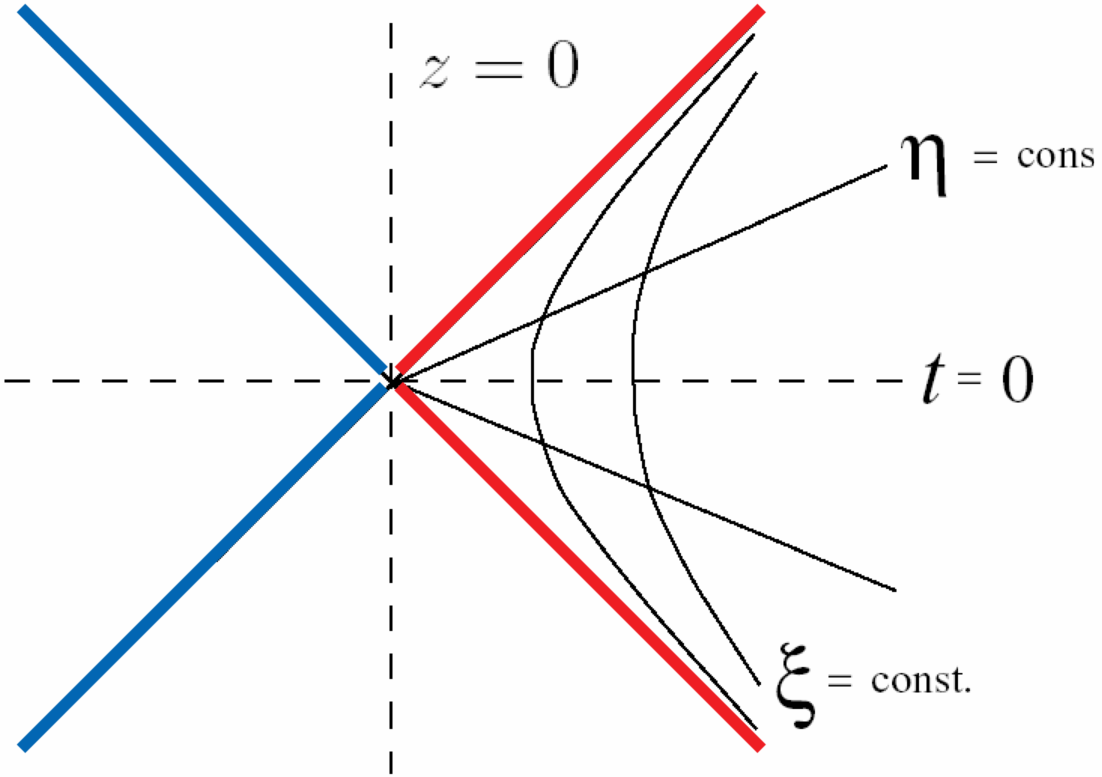
- Entanglement &
  area thermodynamics of Rindler space
- Entanglement & area
- Entanglement & dimensional reduction
  (holography)

Thermodynamics, Area, Holography

- Black Holes $S = \frac{A}{4G_N}$ Bekenstein, Hawking
- Entropy Bounds
  - BEB $S \leq \frac{2\pi}{\hbar c} ER$ Bekenstein
  - Holographic $S \leq \frac{A_{LS}}{4G_N}$ Fichler & Susskind, Bousso
  - Causal $S \leq \sqrt{\frac{EV}{\hbar c G_N}}$ Brustein & Veneziano
- Holographic principle: ‘thooft, Susskind

Boundary theory with a limited #DOF/planck area

Rindler space



$$ds^2 = -dt^2 + dz^2 + d\vec{x}_\perp{}^2$$

$$t(\xi,\eta) = \frac{1}{a} e^{a\xi} \sinh a\eta$$

$$z(\xi,\eta) = \frac{1}{a} e^{a\xi} \cosh a\eta$$

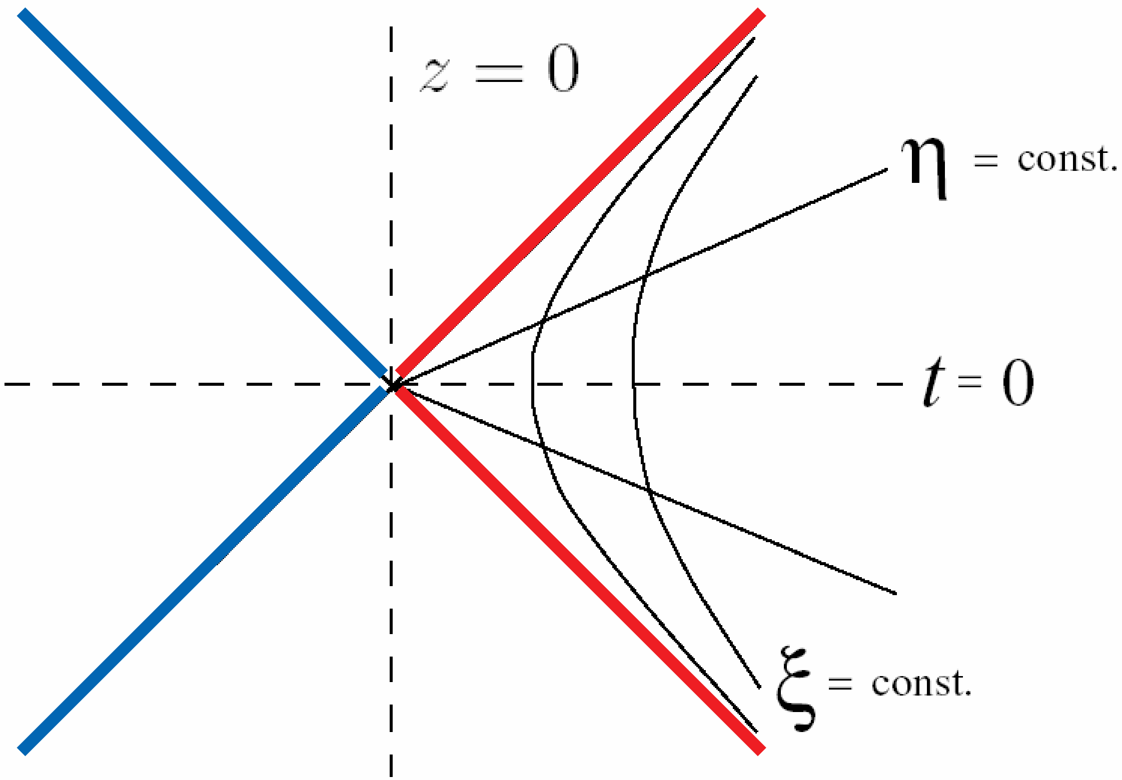
$$\vec{x}_\perp = \vec{x}_\perp .$$

$-\infty < \xi < \infty$ $\quad$ $0 < z$

$-\infty < \eta < \infty$ $\quad$ $t < |z|$

$$ds^2 = -e^{2a\xi} d\eta^2 + e^{2a\xi} d\xi^2 + d\vec{x}_\perp{}^2$$

$$ds^2 = -(aR)^2 d\eta^2 + dR^2 + (dx_\perp)^2$$



$$t(\xi,\eta) = \frac{1}{a}e^{a\xi}\sinh a\eta$$
$$z(\xi,\eta) = \frac{1}{a}e^{a\xi}\cosh a\eta$$
$$\vec{x}_\perp = \vec{x}_\perp.$$

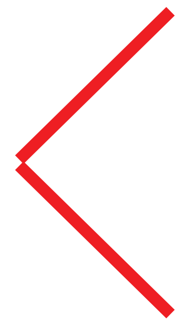
Lines of constant ξ - constant acceleration

Addition of velocities in SR

$$w = \frac{u+v}{1+uv}$$

$$v = \tanh r \Rightarrow t = s + r$$

$$\frac{dv}{d\tau} = A \Rightarrow r = A\tau \Rightarrow v = \tanh A\tau$$

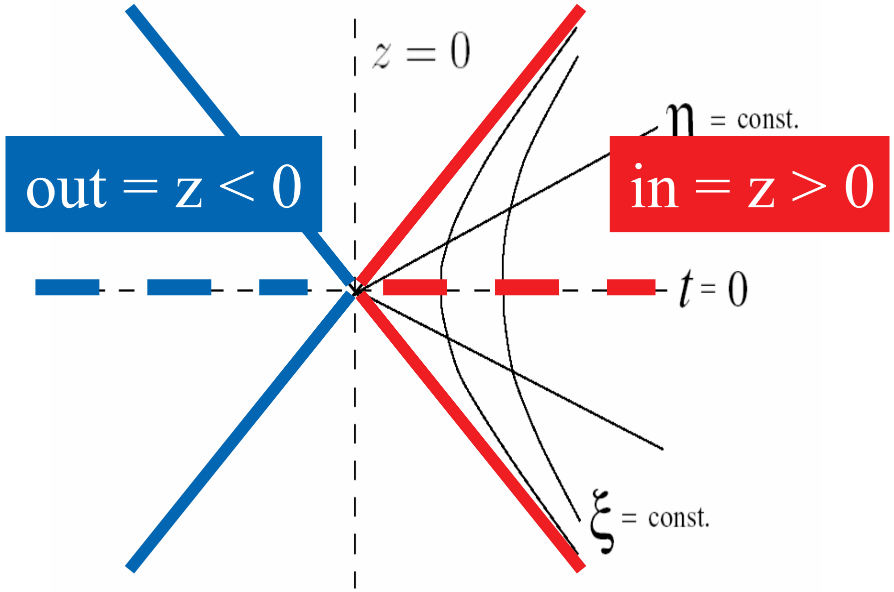
horizon 

$$d\tau^2 = e^{2a\xi}d\eta^2$$

$$v = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \tanh(ae^{-a\xi}\tau) \Rightarrow ae^{-a\xi}$$ proper acceleration

Minkowski vacuum is a Rindler thermal state (Unruh effect)

TFD



$$ds_E^2 = (aR)^2 d\eta^2 + dR^2 + (dx_\perp)^2$$

$$\rho_{in/out} = Tr_{out/in}\rho$$

$$|\psi\rangle = |\psi_{in}\psi_{out}\rangle$$

Compare two expressions for ρ_{in} (by writing them as a PI)

1. $< \psi'_{in}|\rho_{in}|\psi''_{in} >= \int < \psi_{out}\psi'_{in}|0 >< 0|\psi''_{in}\psi_{out} > D\psi_{out}$

2. $< \psi'_{in}|\rho_{in}|\psi''_{in} >= < \psi'_{in}|e^{-\beta_0 H_{eff}}|\psi''_{in} >$

1. $$< \psi'_{in}|\rho_{in}|\psi''_{in} > = \int < \psi_{out}\psi'_{in}|0 >< 0|\psi''_{in}\psi_{out} > D\psi_{out}$$

$$< 0|\psi(\vec{x}) > = \int_{\varphi(\vec{x},0)=\psi(\vec{x})} \exp\left[-\int_0^\infty \dots \int \mathcal{L} d^d x d\tau\right] D\varphi$$

$$< 0|\psi_{in}\psi_{out} > = \int \exp\left[-\int_0^\infty \dots \int \mathcal{L} d^d x d\tau\right] D\varphi$$

$$\varphi(\vec{x},0) = \begin{cases} z > 0 & \psi_{in}(\vec{x}) \\ z < 0 & \psi_{out}(\vec{x}) \end{cases}$$

In general: $$\varphi(\vec{x},0) = \begin{cases} \vec{x} \in in & \psi_{in}(\vec{x}) \\ \vec{x} \in out & \psi_{out}(\vec{x}) \end{cases}$$

$$\int < \psi_{out}\psi'_{in}|0 >< 0|\psi''_{in}\psi_{out} > D\psi_{out} =$$

$$\int\int_{\varphi(\vec{x},0)=\begin{cases} z>0 & \psi'_{in}(\vec{x}) \\ z<0 & \psi_{out}(\vec{x}) \end{cases}} \exp\left[-\int_{-\infty}^{0}\dots\int \mathcal{L}d^d x d\tau\right] D\varphi$$

$$\times \int_{\varphi'(\vec{x},0)=\begin{cases} z>0 & \psi''_{in}(\vec{x}) \\ z<0 & \psi_{out}(\vec{x}) \end{cases}} \exp\left[-\int_{0}^{\infty}\dots\int \mathcal{L}d^d x d\tau\right] D\varphi'\, D\psi_{out}.$$

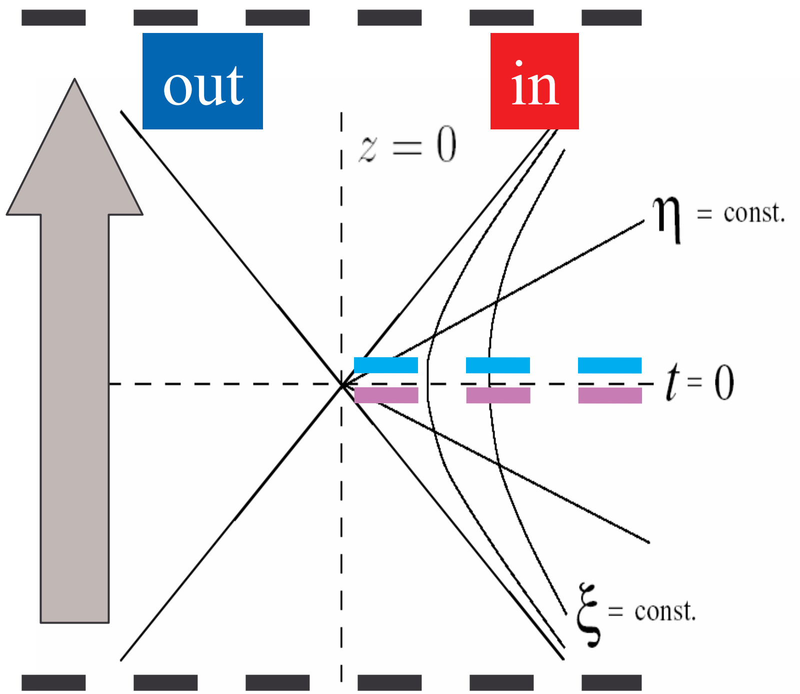
Combining the integrals,

$$= \int\int_{\varphi(\vec{x},0)=\begin{cases} z>0, t=0^- & \psi'_{in}(\vec{x}) \\ z>0, t=0^+ & \psi''_{in}(\vec{x}) \\ z<0 & \psi_{out}(\vec{x}) \end{cases}} \exp\left[-\int_{-\infty}^{\infty}\dots\int \mathcal{L}d^d x d\tau\right] D\varphi D\psi_{out}$$

$$\propto \int_{\varphi(\vec{x},0)=\begin{cases} z>0, t=0^- & \psi'_{in}(\vec{x}) \\ z>0, t=0^+ & \psi''_{in}(\vec{x}) \end{cases}} \exp\left[-\int\dots\int_{-\infty}^{\infty} \mathcal{L}d\tau d^d x\right] D\varphi.$$

Result

$$< \psi'_{in}|\rho_{in}|\psi''_{in} >= \int_{\varphi(\vec{x},0)=\begin{cases} z>0, t=0^- & \psi'_{in}(\vec{x}) \\ z>0, t=0^+ & \psi''_{in}(\vec{x}) \end{cases}} \exp\left[-\int_{-\infty}^{\infty}\cdots\int \mathcal{L} d^d x\, d\tau\right] D\varphi$$



2.

H_{eff} – generator of time translations

Time slicing the interval $[0,\beta_0]$:

$$< \psi'_{in}|e^{-\beta_0 H_{eff}}|\psi''_{in} >$$

$$= \int_{\substack{\varphi(x,0)=\psi'_{in}(\vec{x}) \\ \varphi(x,-i\beta_0)=\psi''_{in}(\vec{x})}} \exp\left[\imath \int_0^{\imath\beta_0} \left(\int \pi \frac{d\varphi}{dt} d^d x - H_{eff} \right) dt \right] D\varphi D\pi$$

$$= \int_{\substack{\varphi(x,0)=\psi'_{in}(\vec{x}) \\ \varphi(x,\beta_0)=\psi''_{in}(\vec{x})}} \exp\left[\int_0^{\beta_0} \left(\int \imath\pi\dot{\varphi} d^d x - H_{eff} \right) (d\tau) \right] D\varphi D\pi$$

Guess:

H_{eff} is the Legendre transform of a free field Lagrangian

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$$

$$H_{eff} = \int \mathcal{H}\sqrt{h}d^dx$$

$$\mathcal{H} = \sqrt{-g_{00}}\left(\frac{1}{2}\frac{\pi}{\sqrt{h}}^2 + \frac{1}{2}h^{ij}\partial_i\varphi\partial_j\varphi\right) \qquad \pi = \frac{\dot{\varphi}\sqrt{h}}{\sqrt{-g_{00}}}$$

result

$$< \psi'_{in}|e^{-\beta_0 H_{eff}}|\psi''_{in} >= \int_{\substack{\varphi(\xi,0)=\psi'_{in}(\xi)\\ \varphi(\xi,\beta_0)=\psi''_{in}(\xi)}} \exp\left[-\int_0^{\beta_0}\int \mathcal{L}_E d^d\xi d\eta\right] |\Omega|\ |g|^{\frac{1}{4}}\ D\varphi$$

$$\Omega = \frac{1}{\sqrt{-g_{00}}}$$

Results

$$< \psi'_{in} | \rho_{in} | \psi''_{in} > = \int_{\varphi(\vec{x},0) = \begin{cases} z>0, t=0^- & \psi'_{in}(\vec{x}) \\ z>0, t=0^+ & \psi''_{in}(\vec{x}) \end{cases}} \exp\left[- \int_{-\infty}^{\infty} \cdots \int \mathcal{L} d^d x d\tau \right] D\varphi$$

$$< \psi'_{in} | e^{-\beta_0 H_{eff}} | \psi''_{in} > = \int_{\substack{\varphi(\xi,0)=\psi'_{in}(\xi) \\ \varphi(\xi,\beta_0)=\psi''_{in}(\xi)}} \exp\left[- \int_0^{\beta_0} \int \mathcal{L}\_ \, d^d \xi d\eta \right] |\Omega| \; |g|^{\frac{1}{4}} D\varphi$$

If

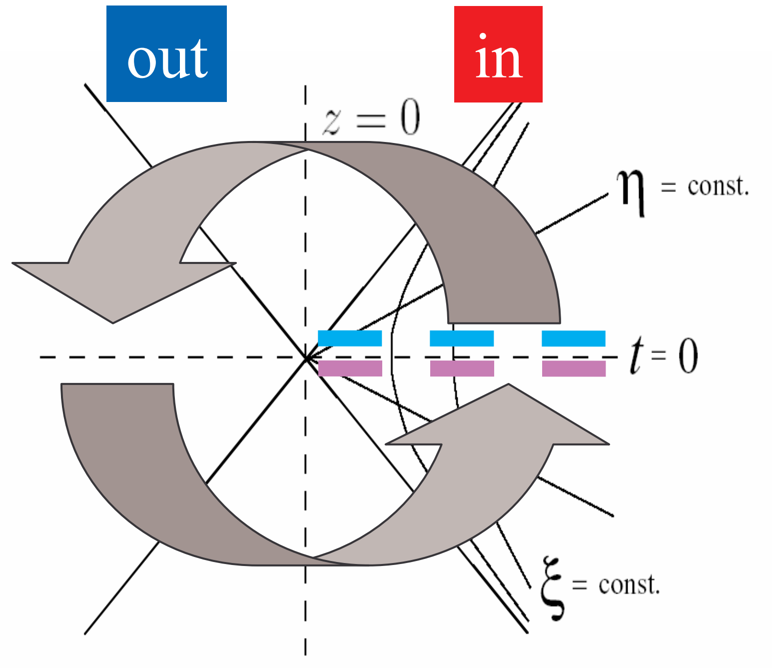
1. The boundary conditions are the same
2. The actions are equal
3. The measures are equal

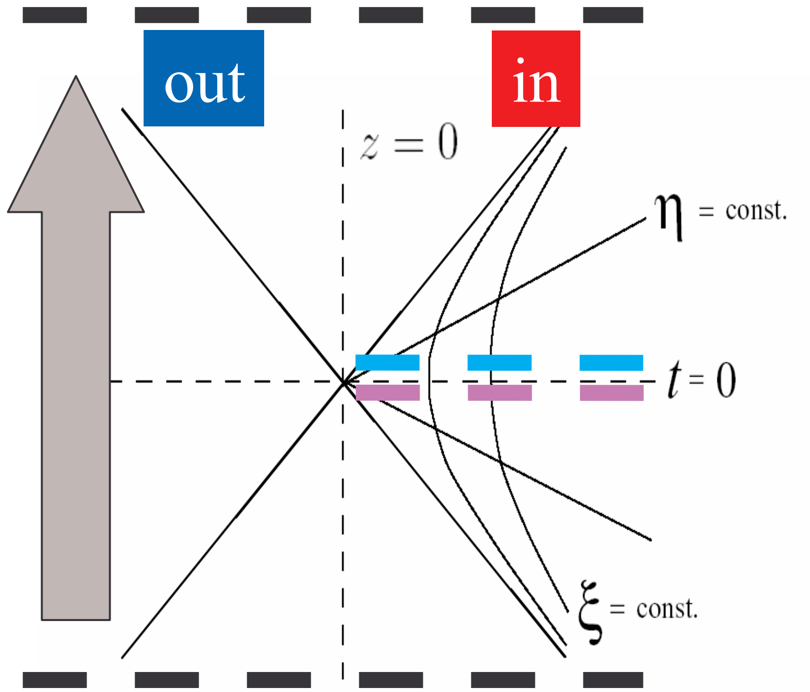
Then $\rho_{in} = e^{-\beta_0 H_{eff}}$

For half space $H_{\text{eff}} = H_{\text{Rindler}}$, $\beta_0 = \frac{2\pi}{a}$

$H_{\text{Rindler}} = \text{boost}$

$$H_R = \frac{1}{2}\int (\partial_\eta \varphi)^2 + (\partial_\xi \varphi)^2 + e^{2a\xi}(\partial_{\vec{x}_\perp}\varphi)^2 d\xi d\vec{x}_\perp$$





Rindler area thermodynamics

Susskind Uglum
Callan Wilczek
Kabat Strassler
De Alwis Ohta
Emparan
…

$$S_\phi = \int_0^\beta dt d^{D-1}x \sqrt{g}\phi(K + m^2)\phi$$

$$K \equiv -\Box + \frac{1}{4}\frac{D-2}{D-1}R \qquad \Box \equiv \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)$$

$$\mathrm{Tr}[e^{-\beta H}] = \int_{\phi(0,\vec{x})=\phi(\beta,\vec{x})} \prod_{t,\vec{x}} d\phi \Omega g^{\frac{1}{4}}(t,\vec{x}) e^{-\int_0^\beta dt d^{D-1}x\sqrt{g}\phi(K+m^2)\phi}$$

$$= \int_{\phi(0,\vec{x})=\phi(\beta,\vec{x})} \prod_{t,\vec{x}} d\phi g^{\frac{1}{4}}(t,\vec{x}) e^{-\int_0^\beta dt d^{D-1}x\sqrt{g}\phi(K+m^2)\phi + S_L[g,\Omega]}$$

$$-\beta F = -\frac{1}{2}\ln\det[K_\beta + m^2] + \beta\int d^{D-1}x\sqrt{g}L_L[\Omega, g]$$

Go to "optical" space $\bar{d}s^2 = dt^2 + \frac{h_{ij}}{g_{00}} dx^i dx^j$

Compute using heat kernel method

$$F(\beta) = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^{\frac{D}{2}}} \sum_{n \neq 0} e^{-\frac{\beta^2 n^2}{4s}} \sum_{k=0}^\infty \frac{(-s)^k}{k!} \bar{B}_k.$$

High temperature approximation

$$F = -T^D V_{D-1} \int_0^\infty \frac{du}{u} \frac{1}{(4\pi u)^{\frac{D}{2}}} \sum_{n=1}^\infty e^{-\frac{n^2}{4u}}$$

$$= -\frac{T^D V_{D-1}}{\pi^{\frac{D}{2}}} \Gamma\left(\frac{D}{2}\right) \zeta(D),$$

In 4D:

$$F = -V_3 T^4 \frac{\pi^2}{90}$$

Volume of optical space $V_{D-1} = \int_{\mathcal{M}_{D-1}} \sqrt{g}$

Compute: $V_{D-1} = \int_{\mathcal{M}_{\mathrm{D}-1}} \sqrt{\bar{g}}$

Euclidean Rindler $ds^2 = R^2 d\omega^2 + dR^2 + (dx_\perp)^2$

Optical metric $\bar{d}s^2 = dt^2 + \frac{h_{ij}}{g_{00}} dx^i dx^j$

$$V_{D-1} = V_{D-2} \int_\epsilon^\infty \frac{dR}{R^{D-1}} = \frac{V_{D-2}}{(D-2)\epsilon^{D-2}} \qquad V_{D-2} \equiv V_\perp$$

$$R_{\min} = \varepsilon$$

$$F = -\frac{\Gamma(D/2)\zeta(D)}{(D-2)\pi^{D/2}} T^D \frac{V_{D-2}}{\varepsilon^{D-2}} \qquad S = -\frac{\partial F}{\partial T}\Big|_{T=T_H} \propto \frac{V_{D-2}}{\varepsilon^{D-2}}$$

In 4D $F = -\frac{A}{\epsilon^2} T^4 \frac{\pi^2}{180}$ $\qquad C_V = \frac{\partial U}{\partial T}\Big|_{T=T_H} \propto \frac{V_{D-2}}{\varepsilon^{D-2}}$

$$\langle \psi | O_{in} | \psi \rangle = Tr(\rho_{in} O_{in})$$

$$H = H_1 \otimes H_2$$

$$|\psi> = \sum_{\alpha a} A_{\alpha a} |a> \otimes |\alpha>$$

$$O_2 = I \otimes O$$

$$O_1 = O \otimes I$$

$$\rho_1 = Tr_2 \rho$$

$$\rho_2 = Tr_1 \rho$$

הפוך

$$O_1|o> = o|o>$$
$$O_2|\theta> = \theta|\theta>$$

$$|o> \quad = \sum_a T_{oa}|a>$$
$$|\theta> \quad = \sum_\alpha S_{\theta\alpha}|\alpha>$$

$$|a> \quad = \sum_o T^\dagger_{ao}|o>$$
$$|\alpha> \quad = \sum_\theta S^\dagger_{\alpha\theta}|\theta>$$

S,T unitary

$$|\psi> = \sum_{\substack{a,\alpha \\ o,\theta}} A_{\alpha a} T^\dagger_{ao} H^\dagger_{\alpha\theta}|o>|\theta>$$
$$= \sum_{\theta,o} (S^\star A T^\dagger)_{\theta o}|o>|\theta> \quad = \sum_{\theta,o} M_{\theta o}|o>|\theta>$$

$$
\begin{aligned}
<\psi|O_1|\psi> &= \sum_{\theta,o} M^{\dagger}_{o\theta} < o| < \theta| O \sum_{\theta',o'} M_{\theta' o'} |o'> |\theta'> \\
&= \sum_{\theta,o} M^{\dagger}_{o\theta} M_{\theta o}\, o \\
&= \sum_{o} M^{\dagger} M_{oo}\, o \\
&= \sum_{o} (T A^{\dagger} S^{T} S^{\star} A T^{\dagger})_{oo}\, o \\
&= \sum_{o} (T A^{\dagger} A T^{\dagger})_{oo}\, o.
\end{aligned}
$$

$$|\psi><\psi| = \sum_{\substack{\alpha,\beta \\ a,b}} |a>|\alpha> A^{\dagger}_{\alpha a} A_{b\beta} < b| < \beta|$$

$$\rho_1 = \sum_{\gamma} <\gamma|\psi><\psi|\gamma>$$

$$= \sum_{\substack{\alpha,\beta,\gamma \\ a,b}} \delta_{\alpha\gamma} |a> A^{\dagger}_{\alpha a} A_{b\beta} < b| \delta_{\beta\gamma} = \sum_{a,b} (AA^{\dagger})_{ba} |a><b|$$

$$O = \sum_{o} |o >< o| o$$

$$\mathrm{Trace}(\rho_1 O) = \sum_{o} < o|\rho_1 O|o >$$

$$= \sum_{o} < o| \left(\sum (AA^{\dagger})_{ba} |a >< b| \sum_{o'} o' |o' >< o'| \right) |o >$$

$$= \sum_{o,b,a} < o|a >< b|o > (AA^{\dagger})_{ba} o \qquad = \sum_{o} T_{oa} T^{\dagger}_{bo} (AA^{\dagger})_{ba}$$

$$= \sum_{o} (TA^{\dagger}AT^{\dagger})_{oo} o = \langle \psi | O | \psi \rangle$$

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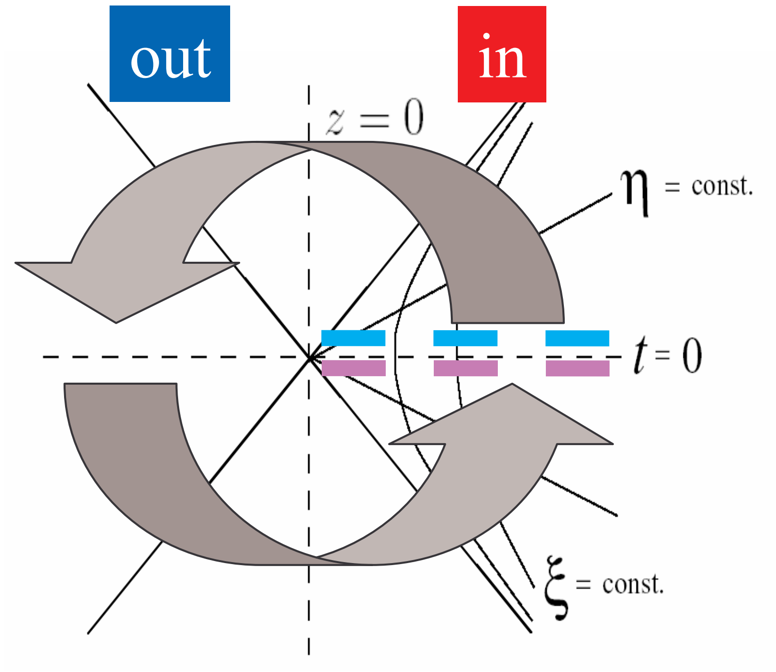
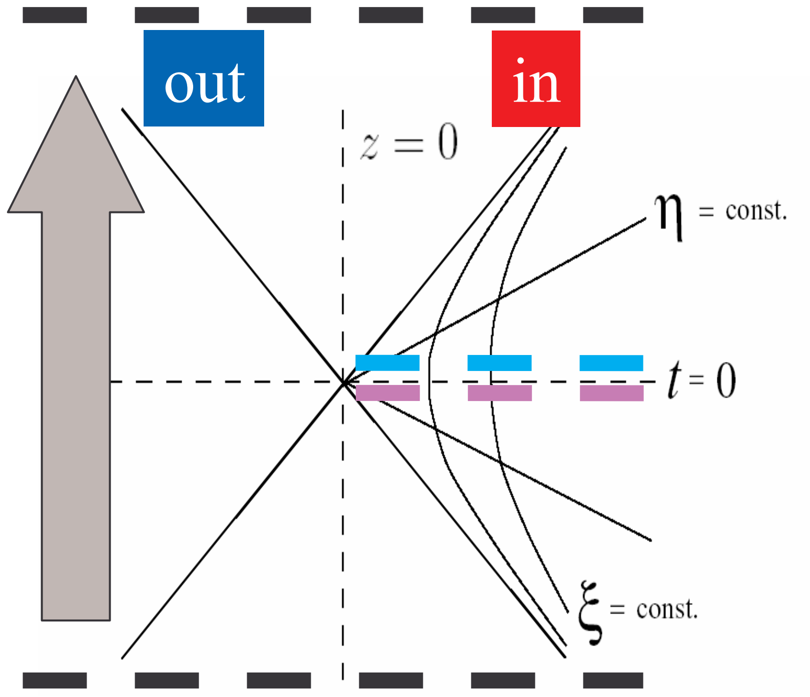
$$\rho_{in} = e^{-\beta_0 H_{eff}}$$

For half space $H_{\mathrm{eff}} = H_{\mathrm{Rindler}}$, $\beta_0 = \dfrac{2\pi}{a}$

$H_{\mathrm{Rindler}} =$ boost





$$\langle \psi | O_{in} | \psi \rangle = Tr(\rho_{in} O_{in})$$

$$H = H_1 \otimes H_2 \qquad |\psi\rangle = \sum_{\alpha a} A_{\alpha a} |a\rangle \otimes |\alpha\rangle$$

$$O_2 = I \otimes O \qquad \rho_1 = Tr_2 \rho$$

$$O_1 = O \otimes I \qquad \rho_2 = Tr_1 \rho$$

הפוך

$(\Delta E^V)^2$

- System in an energy eigenstate

➔ energy does not fluctuate

- Energy of a sub-system fluctuates

➔ "Entanglement energy" fluctuations

Connect to Rindler thermodynamics

For free fields

$$E^V = \int_V :\mathcal{H}(\vec{x}): d^d x \qquad \mathcal{H}(x) = \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\phi(x))^2$$

$$\phi(x) = \int \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{\imath \mathbf{p}\cdot\mathbf{x}} + a^{\dagger}_{-\mathbf{p}} e^{\imath \mathbf{p}\cdot\mathbf{x}}\right) \frac{d^d p}{(2\pi)^d}$$

$$\pi(x) = -\int \imath \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{\imath \mathbf{p}\cdot\mathbf{x}} - a^{\dagger}_{-\mathbf{p}} e^{\imath \mathbf{p}\cdot\mathbf{x}}\right) \frac{d^d p}{(2\pi)^d}$$

$$E^V = \frac{1}{4}\frac{1}{(2\pi)^{2d}} \int \Bigg(\left(\frac{-\vec{p}\cdot\vec{q}}{\sqrt{\omega_p \omega_q}} - \sqrt{\omega_p \omega_q} \right) (a_{\vec{p}} a_{\vec{q}} + a_{-\vec{p}} a_{-\vec{q}}) +$$

$$2 \left(\frac{-\vec{p}\cdot\vec{q}}{\sqrt{\omega_p \omega_q}} + \sqrt{\omega_p \omega_q} \right) a_{-\vec{p}} a_{\vec{q}} \Bigg) e^{\imath(\vec{p}+\vec{q})\cdot\vec{x}} d^d p \, d^d q \, d^d x$$

$$\langle 0 | E^V | 0 \rangle = 0$$

For a massless field

$$\langle 0|(E^V)^2|0\rangle = \frac{1}{8}\frac{1}{(2\pi)^{2d}}\int_V d^d y_1 \int_V d^d y_2$$

$$\times \int d^d p \int d^d q e^{\imath(\vec{p}+\vec{q})\cdot(\vec{y}_1-\vec{y}_2)}\left(\frac{\vec{p}\cdot\vec{q}}{\sqrt{pq}}+\sqrt{pq}\right)^2$$

Vanishes for the whole space!

$$\langle 0|(E^V)^2|0\rangle = \int_0^\infty F(\xi)\, D_V(\xi)\, d\xi$$

$$D_V(\xi) = \int_V \int_V \delta^{(d)}(\xi - |\vec{x}-\vec{y}|)\, d^d x\, d^d y.$$

Geometry

$$F(x) = \frac{1}{8}\frac{1}{(2\pi)^{2d}}\int \left(pq + 2\vec{p}\cdot\vec{q} + \frac{(\vec{p}\cdot\vec{q})^2}{pq}\right) e^{-\imath(\vec{p}+\vec{q})\cdot\vec{x}}\, d^d p\, d^d q$$

Operator

$$F(x) = \frac{1}{8}\frac{1}{(2\pi)^{2d}}\left(\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}(d-1)\right)^2 \quad \vec{x} = \begin{pmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};\ \vec{q} = \begin{pmatrix} q_x \\ q_\perp \\ 0 \\ \vdots \\ 0 \end{pmatrix};\ \vec{p} = \begin{pmatrix} p_x \\ p_\perp \cos\theta_p \\ p_{\perp_\perp} \\ \vdots \\ 0 \end{pmatrix}$$

$$\times \int \left(pq + 2p_x q_x + \frac{p_x^2 q_x^2}{pq} + \frac{p_\perp^2 q_\perp^2}{pq}\frac{1}{d-1}\right)$$

$$\times\, e^{-\imath(p_x+q_x)x} p_\perp^{d-2} q_\perp^{d-2} dp_\perp\, dq_\perp dp_x\, dq_x$$

UV cutoff!!
In this example
Exp*(-p/Λ)*

$$F(x) = \frac{(d+1)\Gamma\left(\frac{d+1}{2}\right)\Lambda^{2(d+1)}}{8\pi^{d+1}(1+(\Lambda x)^2)^{d+3}}(d - 2(d+2)(\Lambda x) + d(\Lambda x)^4$$

$$= \frac{(d+1)\Gamma\left(\frac{d+1}{2}\right)\Lambda^{2(d+1)}}{8\pi^{d+1}}{}^{\prime}\partial^2_{\Lambda x}\frac{(\Lambda x)^2 - 1)}{2(d+2)(1+(\Lambda x)^2)^{d+1}}$$

$\Lambda x >> 1 \quad F(x) \sim x^{-2(d+1)}$

$\Lambda x << 1 \quad F(x) \sim \Lambda^{2(d+1)}$

$$D_V(\xi) = \int_V \int_V \delta^{(d)}(\xi - |\vec{x} - \vec{y}|) d^d x \, d^d y.$$

For half space $\vec{r}_\pm = \vec{x} \pm \vec{y}$ $\quad \vec{r}_\pm = (z_\pm, \vec{r}_{\pm\perp})$

$$D_+(\xi) = \frac{1}{2^d} \underline{V_\perp} \int_0^\infty \int_{-z_+}^{z_+} \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \delta^{(d)}(\xi - r_-) d^{d-1} r_{-\perp} \right] dz_- dz_+$$

$$\langle 0 | \left(E^+ \right)^2 | 0 \rangle =$$

$$\frac{1}{2^d} \underline{V_\perp \Lambda^{d-1}} \int_0^\infty \int_{-z_+}^{z_+} \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F \left(\frac{r_-}{\Lambda} \right) d^{d-1} r_{-\perp} \right] dz_- dz_+$$

$$\langle 0 \| \left(E^+ \right)^2 | 0 \rangle =$$

$$\frac{1}{2^d} V_\perp \Lambda^{d-1} \int_0^\infty \int_{-z_+}^{z_+} \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F \left(\frac{r_-}{\Lambda} \right) d^{d-1} r_{-\perp} \right] dz_- dz_+$$

$$ds^2 = d\rho^2 + dz_-^2 + \rho^{d-2} d\Omega^2$$

$$\langle 0 \| \left(E^+ \right)^2 | 0 \rangle \propto \int_0^\infty \int_{-z_+}^{z_+} \int_0^\infty \Big[\frac{\partial}{\partial \rho} \left(\rho^{d-2} \frac{\partial}{\partial \rho} \left(\frac{\rho^2 + z_-^2 - 1}{2(d+2)(1+\rho^2+z_-^2)^{d+1}} \right) \right) + \rho^{d-2} \frac{\partial^2}{\partial z_-^2} \left(\frac{\rho^2 + z_-^2 - 1}{2(d+2)(1+\rho^2+z_-^2)^{d+1}} \right) \Big] d\rho\, dz_- dz_+$$

$$\langle 0 \| \left(E^+ \right)^2 | 0 \rangle = V_\perp \Lambda^{d+1} \frac{(d+1)\Gamma^2\left(\frac{d+1}{2}\right)}{2^{2d+2} \pi^{1+\frac{d}{2}} \Gamma\left(2+\frac{d}{2}\right)}$$

Rindler specific heat

$$\langle 0_M | (H_R)^2 | 0_M \rangle = Tr\left(e^{-\beta H_R} (H_R)^2\right) = T^2 C_V$$

$$\begin{aligned} H_R &= \int \sqrt{h}\sqrt{-g_{00}} \left[\frac{1}{2}\pi_{dA}^2 + \frac{1}{2} h^{ij}\partial_i\varphi\partial_j\varphi\right] d\xi d\vec{x}_\perp \\ &= \int e^{a\xi} e^{a\xi} \left[\frac{1}{2}(\partial_\eta\varphi)^2 e^{-2a\xi} + \frac{1}{2}\left((\partial_\xi\varphi)^2 e^{-2a\xi} + (\partial_{\vec{x}_\perp}\varphi)^2\right)\right] d\xi d\vec{x}_\perp \\ &= \frac{1}{2}\int (\partial_\eta\varphi)^2 + (\partial_\xi\varphi)^2 + e^{2a\xi}(\partial_{\vec{x}_\perp}\varphi)^2 d\xi d\vec{x}_\perp. \end{aligned}$$

@ η=0

$$\begin{aligned} H_R &= \frac{1}{2}\int e^{a\xi}\left[(\pi)^2 + (\partial_z\varphi)^2 + (\partial_{\vec{x}_\perp})^2\right] dz d\vec{x}_\perp. \\ &= \frac{1}{2}\int (az)\left[(\pi)^2 + (\partial_z\varphi)^2 + (\partial_{\vec{x}_\perp})^2\right] dz d\vec{x}_\perp. \end{aligned}$$

$$H_R = \frac{1}{2}\int e^{a\xi}\left[(\pi)^2 + (\partial_z\varphi)^2 + (\partial_{\vec{x}_\perp})^2\right] dz d\vec{x}_\perp.$$
$$= \frac{1}{2}\int (az)\left[(\pi)^2 + (\partial_z\varphi)^2 + (\partial_{\vec{x}_\perp})^2\right] dz d\vec{x}_\perp.$$

$$a^{-2}\langle 0_M | (:H_R:)^2 | 0_M \rangle = \int z_1 z_2 F_d(r_-) d^d x_1\, d^d x_2$$

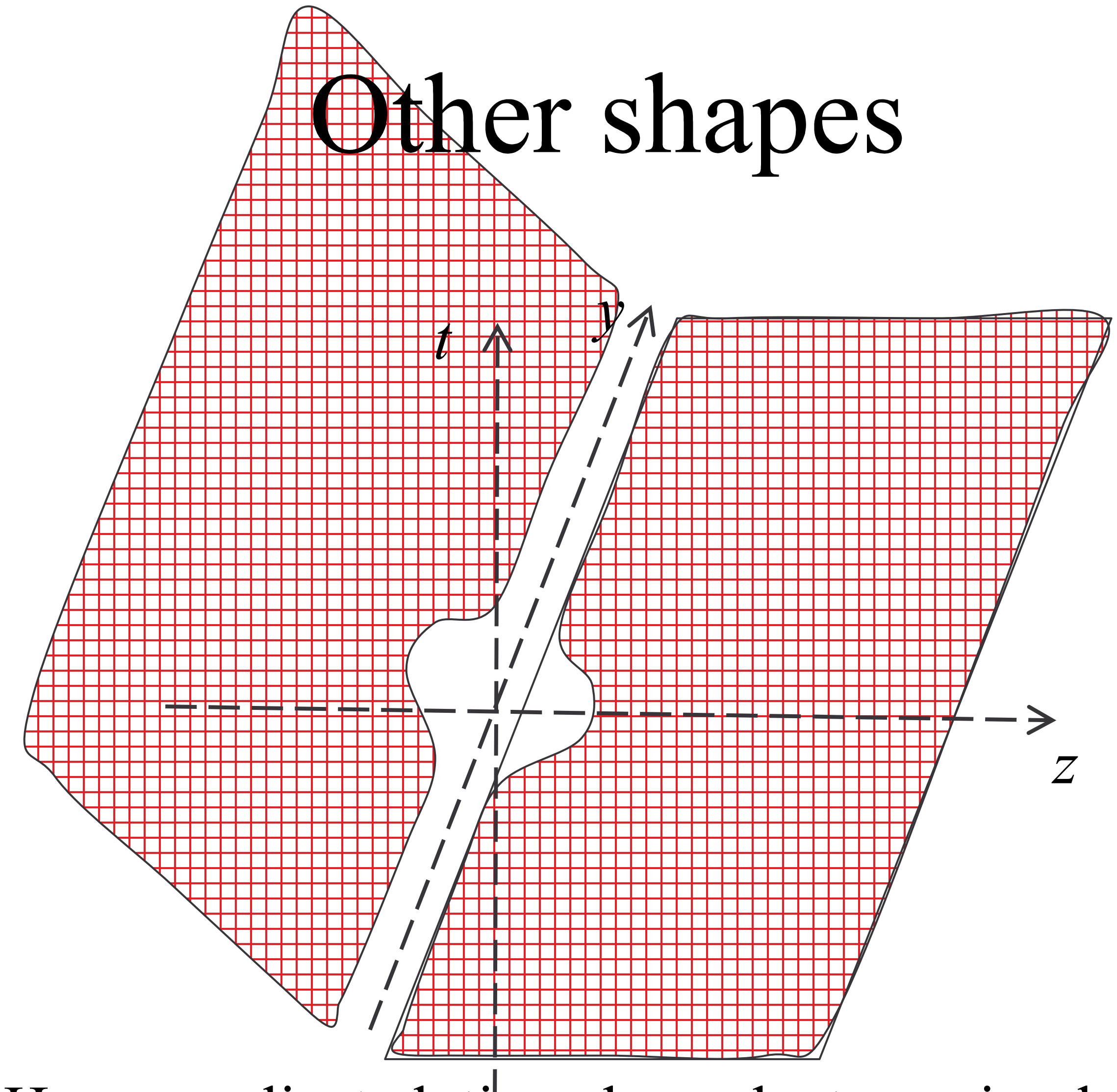
$$= \frac{1}{2^d} V_\perp \int_0^\infty \int_{-z_+}^{z_+} \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{4}(z_+^2 - z_-^2) F_d(r_-) d^{d-1} r_{-\perp} \right] dz_-\, dz_+$$

$$\langle 0_M | (:H_R:)^2 | 0_M \rangle = \frac{a^2}{(d-1)\Lambda^2} \langle 0_M | (E^+)^2 | 0_M \rangle =$$

$$= V_\perp \Lambda^{d-1} a^2 \frac{(d+1)\Gamma^2\left(\frac{d+1}{2}\right)}{(d-1) 2^{2d+2} \pi^{\frac{d+2}{2}} \Gamma\left(2 + \frac{d}{2}\right)}$$

$E^+ = \ldots$ ➔ contributions from the near horizon region

Other shapes



H_{eff} complicated, time dependent, no simple thermodynamics, area dependence o.k.

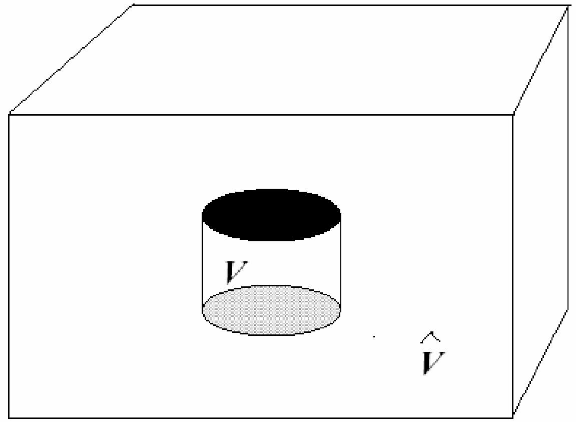
For area thermodynamics need – Thermofield double

Entanglement and area

$$O_i = \int_{V+\hat{V}} d^d x \mathcal{O}_i(\vec{x})$$

$$O_i = O_i^V + O_i^{\hat{V}}$$

$$O_j|0\rangle = 0$$



|0> is not necessarily an eigenstate of $O_j^{\hat{V}}$, O_j^V

|0> is an entnangled state w.r.t. V

Show: $\langle 0|O_i^V O_j^V|0\rangle = \langle 0|O_i^{\hat{V}} O_j^{\hat{V}}|0\rangle$

Non-extensive!, depends on boundary
(similar to entanglement entropy)

Proof:

$$\begin{aligned} 0 &= \langle 0| \left(O_i^V - O_i^{\widehat{V}}\right)\left(O_j^V + O_j^{\widehat{V}}\right)|0\rangle \\ &= \langle 0|O_i^V O_j^V - O_i^{\widehat{V}} O_j^{\widehat{V}} + O_i^V O_j^{\widehat{V}} - O_i^{\widehat{V}} O_j^V|0\rangle \end{aligned}$$

$$\langle 0|\mathcal{O}_i(\vec{y}_1)\mathcal{O}_j(\vec{y}_2)|0\rangle = f_{ij}\left(|\vec{y}_1 - \vec{y}_2|\right)$$

$$\langle 0|O_i^V O_j^{\widehat{V}} - O_i^{\widehat{V}} O_j^V|0\rangle = \int\limits_V d^d y_1 \int\limits_{\widehat{V}} d^d y_2 \left(f_{ij}\left(|\vec{y}_1 - \vec{y}_2|\right) - f_{ij}\left(|\vec{y}_2 - \vec{y}_1|\right)\right) = 0$$

Show that $\langle 0|O_i^V O_j^V|0\rangle$ is linear in boundary area

$$\langle 0|O_i^V O_j^V|0\rangle = \int_V d^d y_1 \int_V d^d y_2 \; f_{ij}\left(|\vec{y}_1 - \vec{y}_2|\right)$$

$$\mathcal{D}_V(y) \sim y^{d-1} \text{ for } y \sim 0$$

$$\mathcal{D}_V(y) = y^{d-1} R^d \widetilde{\mathcal{D}}_V\left(\tfrac{y}{R}\right)$$

$$D_V(2R) = 0$$

$$\widetilde{\mathcal{D}}'_V(0) \neq 0$$

R is the radius of the smallest sphere containing V

$$f_{ij}\left(|\vec{x}|\right) \sim \frac{1}{|\vec{x}|^{\delta_i+\delta_j}}$$

$$\widetilde{f}_{ij}\left(|\vec{q}|\right) \sim |\vec{q}|^{\delta_i+\delta_j-d}$$

$$\int_0^{2R} dy\; \mathcal{D}_V(y)\; f_{ij}(y) = R^d \int_0^{2R} dy\; y^{d-1}\; \widetilde{\mathcal{D}}_V\left(\frac{y}{R}\right) \int \frac{d^d q}{(2\pi)^d}\; e^{-iqy\cos\theta_q}\, \widetilde{f}_{ij}(|\vec{q}|)$$

Need to evaluate

$$I_n = R^d \int_0^{2R} dy\, y^{d-1}\, \widetilde{\mathcal{D}}_V\left(\frac{y}{R}\right) \int d^d k\, e^{iky\cos\theta_k - \frac{1}{2}k^2/\Lambda^2} (k^2)^n$$

$$-k^2 = \frac{1}{y^{d-1}} \partial_y y^{d-1} \partial_y$$

$$I_n = (-1)^n\, a_V\, C_d\, R^{d-1} \Lambda^{2n-1}$$

$$C_d \sim \int_0^\infty dx x^{d-2} \left[\left(\frac{1}{x^{d-1}} \partial_x x^{d-1} \partial_x \right)^{n-1} e^{-\frac{1}{2}x^2} \right]$$

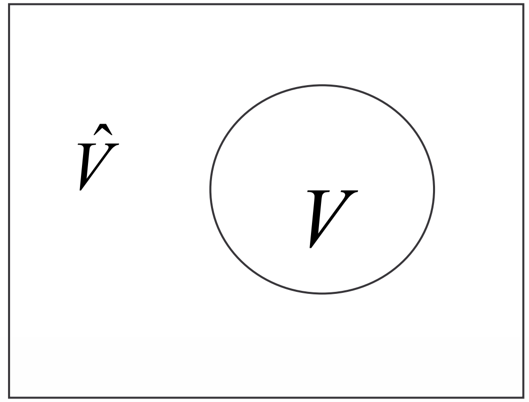
$$a_V = (2\pi)^{d/2}\, \widetilde{\mathcal{D}}'_V(0)$$

✓ $I_\alpha \Leftrightarrow k^\alpha$

✓ General cutoff

Numerical factors depend on regularization

$(\Delta E^V)^2$ for a d-dimensional sphere



$$F(x) = \frac{(d+1)\Gamma\left(\frac{d+1}{2}\right)\Lambda^{2(d+1)}}{8\pi^{d+1}(1+(\Lambda x)^2)^{d+3}}\left(d - 2(d+2)(\Lambda x) + d(\Lambda x)^4\right)$$

$$D_V(x) = \frac{(C(d)d)^2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}\int_{|\alpha|<1} x(r_+^2 - r_-^2)\left[(x^2 - r_+^2)(r_-^2 - x^2)\right]^{\frac{d-3}{2}} dr_+\, dr_-$$

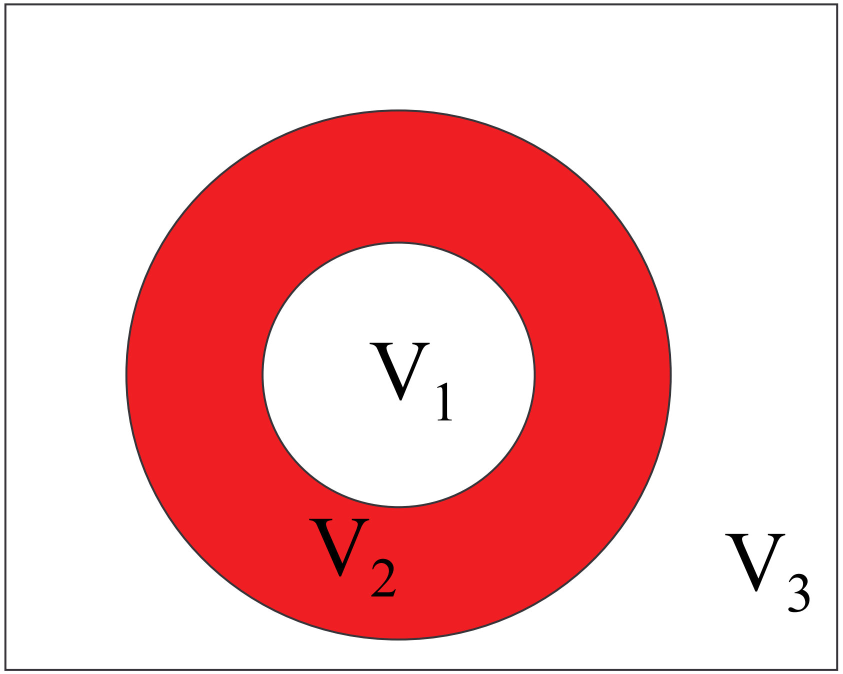
$$\alpha = \frac{x^2 - r_1^2 - r_2^2}{2r_1 r_2}$$

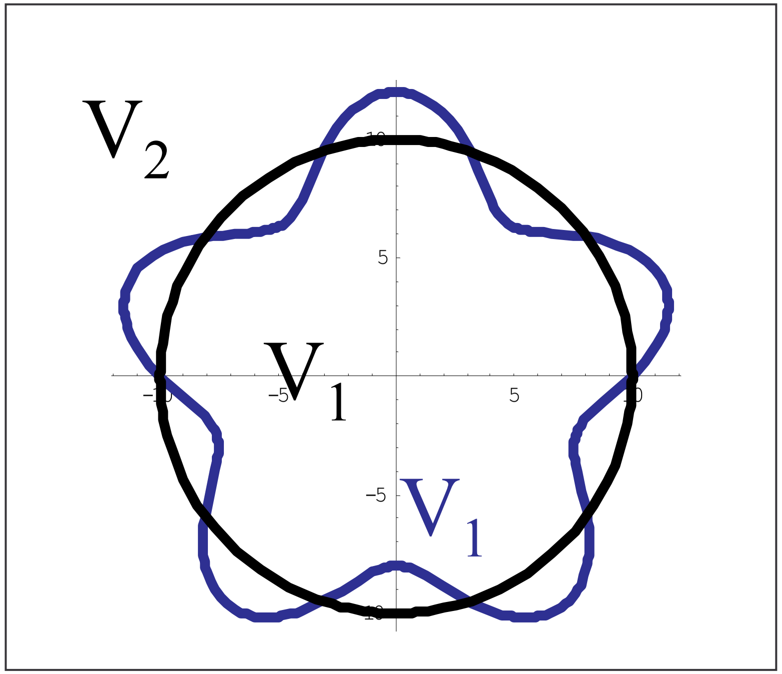
$$\langle 0|(E^V)^2|0\rangle = \frac{K_d}{2^{d+1}\pi^2}\Lambda^{d+1}R^{d-1}$$

K_d	d
$\frac{8}{15}$	3
$\frac{64}{105}$	5
$\frac{1024}{1575}$	7
$\frac{16384}{24255}$	9
$\frac{131072}{189189}$	11

$$K_{27} = 140737488355328/189060384200625$$

Fluctuations live on the boundary



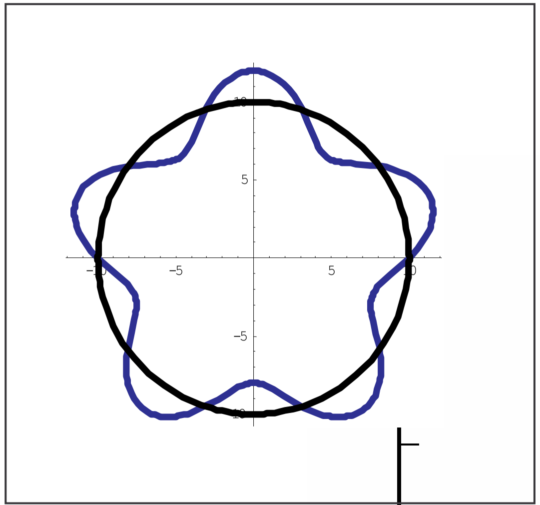


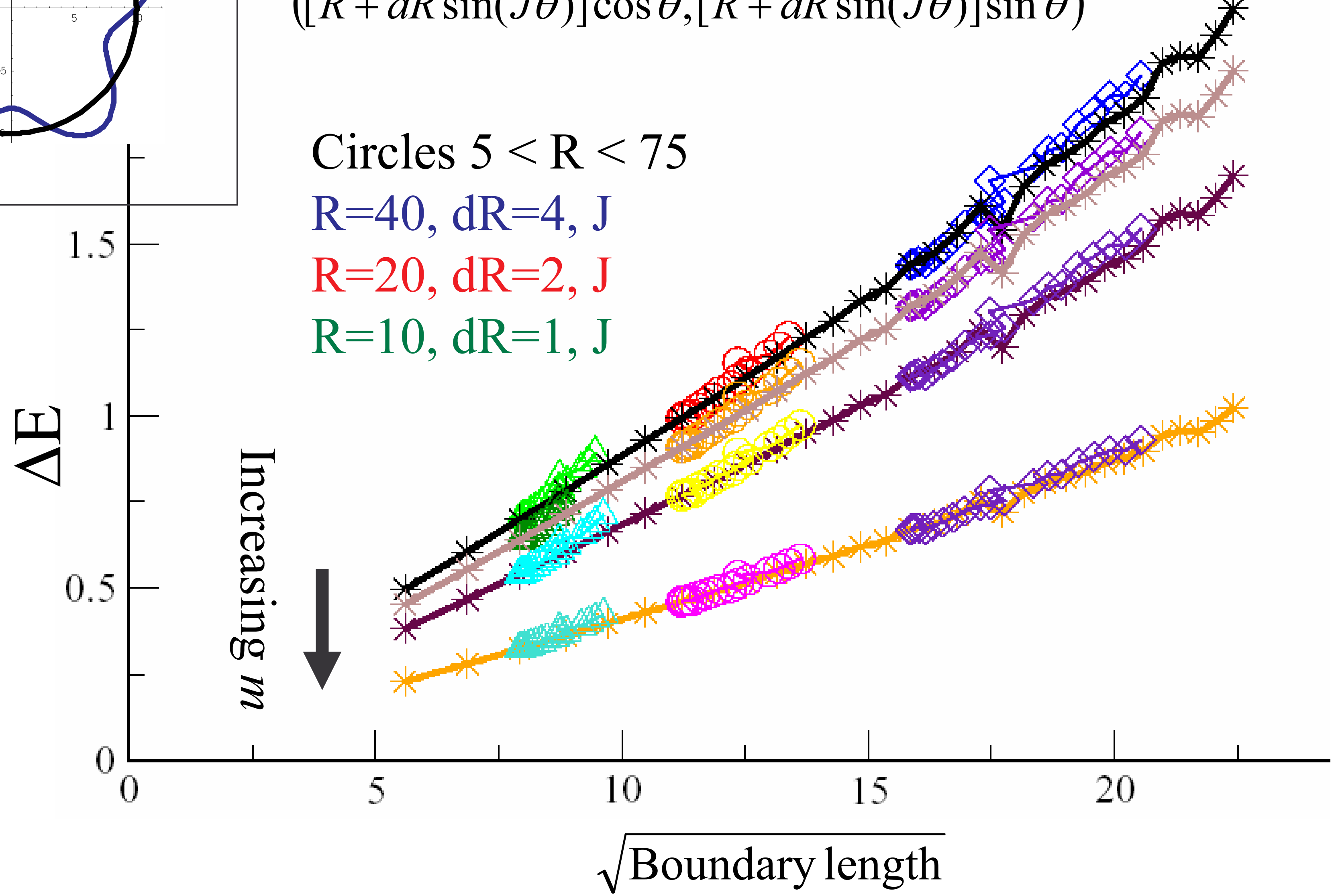
Covariance

$$([R + dR\sin(J\theta)]\cos\theta, [R + dR\sin(J\theta)]\sin\theta)$$

The “flower”

$$([R + dR\sin(J\theta)]\cos\theta, [R + dR\sin(J\theta)]\sin\theta)$$





Boundary theory ?

Express $\langle 0|O_i^V O_j^V|0\rangle$ as a double derivative and convert to a boundary expression

This is possible iff $\widetilde{f}_{ij}(0) = 0$ $\left(\frac{\widetilde{f}_{ij}(q^2)}{q^2} \xrightarrow[q^2 \to 0]{} 0 \right)$

which is generally true for operators of interest

$$f_{ij}\left(|\vec{x}|\right) \sim \frac{1}{|\vec{x}|^{\delta_i+\delta_j}} \qquad \widetilde{f}_{ij}\left(|\vec{q}|\right) \sim |\vec{q}|^{\delta_i+\delta_j-d}$$

$$\langle 0|O_i^V O_j^V|0\rangle \sim \int_V d^d y_1 \int_V d^d y_2 \; \frac{c_{ij}^d}{|\vec{y}_1 - \vec{y}_2|^{\delta_i+\delta_j}}$$

$$= \alpha_{ij}^d \int_V d^d y_1 \int_V d^d y_2 \; \vec{\nabla}_1 \cdot \vec{\nabla}_2 \frac{c_{ij}^{d-1}}{|\vec{y}_1 - \vec{y}_2|^{\delta_i+\delta_j-2}}$$

$$= \alpha_{ij}^d \oint_{\partial V} d^{d-1} z_1 \oint_{\partial V} d^{d-1} z_2 \; \frac{c_{ij}^{d-1} \hat{n}_1 \cdot \hat{n}_2}{|\vec{z}_1 - \vec{z}_2|^{\delta_i+\delta_j-2}},$$

$$\alpha_{ij}^d = \frac{c_{ij}^d}{(\delta_i+\delta_j-2)(d-\delta_i-\delta_j)c_{ij}^{d-1}}$$

$\delta_i+\delta_j = 2 \leftrightarrow$ logarithmic

$\delta_i+\delta_j = d \leftrightarrow$ δ-function

Boundary* correlation functions

$$\langle 0|O_i^V O_j^V|0\rangle \sim \langle 0|\Theta_i^{\partial V}\Theta_j^{\partial V}|0\rangle_{d-1} = \alpha_{ij}^d \int_{\partial V} d^{d-1}z_1 \int_{\partial V} d^{d-1}z_2 \; \hat{n}_1 \cdot \hat{n}_2 \langle 0|\vartheta_i(\vec{z}_1)\vartheta_j(\vec{z}_2)|0\rangle_{d-1}$$

$$\langle 0|\vartheta_i(\vec{z}_1)\vartheta_j(\vec{z}_2)|0\rangle_{d-1} \sim \frac{c_{ij}^{d-1}}{|\vec{z}_1-\vec{z}_2|^{\delta_i+\delta_j-2}}.$$

Show (massless free field, V half space, large # of fields N)

$$\langle 0|e^{-J\int_V \phi(\vec{x})d^d x}|0\rangle_d = \left\langle e^{-J\sqrt{\alpha_{\phi\phi}^d}\int_{\partial V}\phi(\vec{x})d^{d-1}x}\right\rangle_{d-1}^{\beta\to 0}$$

First, n-point functions of single fields

$$\int_V d^d x_1 \ldots \int_V d^d x_{2n} \langle 0|\phi(\vec{x}_1)\cdots\phi(\vec{x}_{2n})|0\rangle_d \;=$$

$$(\alpha^d_{\phi\phi})^n \int_{\partial V} d^{d-1}x_1 \cdots \int_{\partial V} d^{d-1}x_{2n} \langle \phi(\vec{x}_1)\cdots\phi(\vec{x}_{2n})\rangle^{\beta\to 0}_{d-1}$$

$$\langle 0|\phi(\vec{x}_1)\cdots\phi(\vec{x}_{2n})|0\rangle \;=\; \sum_{\substack{\text{all}\\ \text{perm.}}} \langle 0|\phi(\vec{x}_{i_1})\phi(\vec{x}_{i_2})|0\rangle \cdots \langle 0|\phi(\vec{x}_{i_{2n-1}})\phi(\vec{x}_{i_{2n}})|0\rangle$$

$$\int_V d^d x_1 \int_V d^d x_2 \langle 0|\phi(\vec{x}_1)\phi(\vec{x}_2)|0\rangle_d \;=$$

$$\delta = \frac{d-1}{2}$$

$$= \alpha^d_{\phi\phi} \int_{\partial V} d^{d-1}x_{1_\perp} \int_{\partial V} d^{d-1}x_{2_\perp} \frac{\hat{n}_1 \cdot \hat{n}_2}{|\vec{x}_1 - \vec{x}_2|^{d-3}}$$

$$= \alpha^d_{\phi\phi} \int_{\partial V} d^{d-1}x_{1_\perp} \int_{\partial V} d^{d-1}x_{2_\perp} \langle \phi(\vec{x}_1)\phi(\vec{x}_2)\rangle^{\beta\to 0}_{d-1}.$$

$$\int_V d^d x_1 \int_V d^d x_2 \langle 0 | \nabla_1^m \phi^n(\vec{x}_1) \nabla_2^{m'} \phi^{n'}(\vec{x}_2) | 0 \rangle_d \qquad \cong$$

$$\int_{\partial V} d^{d-1} x_{1_\perp} \int_{\partial V} d^{d-1} x_{2_\perp} \langle \nabla_1^{m+n-1} \phi^n(\vec{x}_1) \nabla_2^{m'+n-1} \phi^{n'}(\vec{x}_2) \rangle_{d-1}^{\beta \to 0}$$

Then, show that in the large N limit equality holds for all correlation functions

Only contribution in leading order in N comes from

$$\int_V \ldots \int_V < \mathrm{Tr}\Phi(x_1) \ldots \mathrm{Tr}\Phi(x_n) > d^d x_1 \ldots d^d x_n$$

$$\Phi(x_i) = \mathrm{diag}(\phi_1(x_i), \ldots, \phi_N(x_i))$$

Summary

- Entanglement &
  area thermodynamics of Rindler space
- Entanglement & area
- Entanglement & dimensional reduction