Mon. Not. R. Astron. Soc. **000**, 1–13 () Printed 9 March 2010 (MN LaTeX style file v2.2)

Non-exponential hydrodynamical growth in density-stratified thin Keplerian discs

Yu. M. Shtemler[1⋆], M. Mond[1], G. Rüdiger[2], O. Regev[3], O. M. Umurhan[4]
[1] *Department of Mechanical Engineering, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel*
[2] *Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482 Potsdam, Germany*
[3] *Department of Physics, Technion Israel Institute of Technology, 32000 Haifa, Israel*
and Department of Astronomy, Columbia University, New York, New York 10027, USA
[4] *Astronomy Unit, School of Mathematical Sciences, Queen Mary University of London,*
Mile End Road, London E1 4NS, United Kingdom
and Department of Astronomy, City College San Francisco, San Francisco, California 94112, USA

Accepted —. Received —-; in original form —-

ABSTRACT
The short time evolution of three dimensional small perturbations is studied. Exhibiting spectral asymptotic stability, thin discs are nonetheless shown to host intensive hydrodynamical activity in the shape of non modal growth of initial small perturbations. Two mechanisms that lead to such behavior are identified and studied, namely, non-resonant excitation of vertically confined sound waves by stable planar inertia-coriolis modes that results in linear growth with time, as well as resonant coupling of those two modes that leads to a quadratic growth of the initial perturbations. It is further speculated that the non modal growth can give rise to secondary strato-rotational instabilities and thus lead to a new route to turbulence generation in thin discs.

Key words: accretion discs, plasmas, protoplanetary discs

1 INTRODUCTION

Reverting to Rayleigh's stability criterion Keplerian accretion discs have been deemed centrifugally stable. This has lead to focusing the attention on the magneto-rotational instability (MRI) which is excited due to the freezing of magnetic field lines into fluid elements Balbus & Hawley (1991). Indeed, the MRI has been accepted since as a main turbulence generator and has consequently been widely considered to hold the key to the solution of the angular transfer problem in accretion discs. However, the MRI as a driver of turbulence and angular momentum transport has recently been shown to suffer from non-trivial difficulties in the sheer use of the shearing box in the simulations [Regev & Umurhan (2008), Bodo et al. (2008)] as well as in doubts regarding the numerical convergence and resolution [Lesur & Longaretti (2007), Fromang & Papaloizou (2007), Fromang et al. (2007), Pessah et al. (2007)]. In addition, it has recently been shown that if the thin disk geometry is taken into account both the growth rates as well as the number of unstable MRI modes are greatly reduced and are decreasing functions of the disk thickness [Coppi & Keyes (2003), Liverts & Mond (2009)]. In parallel, various works have indicated over the last decade that pure hydrodynamical processes are not as irrelevant as possible sources for enhanced transport coefficients in thin accretion discs as has been hastily assumed before. In particular, two topics have caught the attention of researchers. In the first one the destabilizing effect of a stable stratification on a centrifugally stable rotating fluid has been examined. The resulting instability, termed the strato-rotational instability (SRI), has been investigated for Taylor-Couette flows [Boubnov et al. (1995), Yavneh et al. (2001), Shalybkov & Rüdiger (2005)] or for Taylor-Couette-like flows within the shearing sheet approximation that models rotating discs [Dubrulle et al. (2005), Umurhan (2006), Tevzadze et al. (2008)]. Common to those works is: 1. Rigid-like boundary conditions at two radial locations, 2. Radial pressure distribution plays a crucial role in the onset of the SRI under those conditions, 3. Ignoring the vertical boundaries of the fluid domain, and 4.

⋆ E-mail: shtemler@bgu.ac.il; mond@bgu.ac.il; gruediger@aip.de; regev@astro.columbia.edu; umurhan@maths.qmul.ac.uk

Incompressibility within the Boussinesq approximation. We will return later on to these points (actually 1 and 2 are two sides of the same phenomenon) but let us first move on to the second route that may lead to pure hydrodynamical turbulence. The latter is motivated by successful attempts by fluid dynamics to explain the transition to turbulence in spectrally stable flows [Boberg & Brosa (1988), Grossmann (2000), Schmid & Henningson (2000)]. Thus, rendering the linear operator non normal, the flow shear may give rise to transiently growing perturbations [Gustavsson (1991), Butler & Farrel (1992), Reddy & Henningson (1993), Trefethen et al. (1993), Criminale et al. (1997), and the recent review by Schmid (2007)] whose energy is eventually redistributed among different length-scales due to nonlinear effects. Indeed, recent calculations in real thin disc geometry have demonstrated the efficiency of such processes to significantly amplify initially small perturbations, and thus to generate intensive hydrodynamical activity in the otherwise centrifugally stable Keplerian discs [Umurhan et al. (2006) , Rebusco et al. (2009)]. Such non modal growth of perturbations has also been obtained for ballooning modes in magnetized rotating plasmas in tokamaks [Chun & Hameiri (1990)]. The two routes described above are shown in the current work to diverge from a single unified comprehensive stability analysis in real thin disc geometry. It is thus shown first that if the proper physical conditions as well as the true thin disc geometry are taken into account, no spectrally unstable strato-rotation modes exist. Indeed, the lack of radial rigid boundaries, and the negligible role of the radial pressure gradients in establishing the steady-state rotation stand in stark distinction from the conditions that give rise to the SRI in Couette flows. Furthermore, the small thickness of the disc combined with its steady-state supersonic rotation results in vertical sound crossing time that is of the order of a rotation period. Consequently, compressibility effects cannot be ignored so that vertically propagating sound waves play a major role in the dynamical response of thin discs to small perturbations.

Thus, (temporary) disappearing from the thin discs global scene (they will resurface later on), the SRI leaves the stage entirely to algebraic instabilities as a primary source of hydrodynamical activity. As will be shown there are two mechanisms that give rise to such non-exponential dynamical response. One is the non-resonant coupling between the two modes of wave propagation in the system (the inertia-coriolis waves and the sound waves). Such coupling is a direct result of the rotation shear that renders the linear operator non normal. Due to such interaction, sound waves are non-resonantly driven by the inertia-coriolis modes and their amplitude grows linearly in time. The second source of non-exponential perturbation growth is the resonant interaction between the above mentioned modes. In such scenario, sound waves are resonantly driven by the inertia-coriolis modes and due to the shear their amplitude grows quadratically in time.

Finally, the two routes converge again by the tendency of the algebraically growing perturbations to develop sharp radial pressure gradients due to the rotation shear. As such gradients are essential ingredients in the onset of the SRI, it is conjectured that the latter may be excited as a secondary instability on top of the non modal growth.

The paper is organized as follows. The dimensionless governing equations and their approximation to leading order in small aspect ratio of the steady-state disc are presented in the next Section. Section 3 describes the principal hydrodynamic modes of waves which can propagate in Keplerian discs. Non-resonantly as well as resonantly driven sound waves by inertial waves are considered in Sections 4 and 5, respectively. Conclusions are presented in section 6. Appendix A contains the principal boundary-value problems and methods of their solutions.

2 THE GOVERNING EQUATIONS

The dynamical response of radially and axially stratified thin rotating fluid discs to small perturbations is considered. For that purpose it is convenient to transform all the physical variables to non-dimensional quantities by using the following characteristic values [Shtemler et al. (2007), (2009)]:

$$V_* = \Omega_* r_*,\ t_* = \frac{1}{\Omega_*},\ \Phi_* = {V_*}^2,\ m_* = m_i,\ n_* = n_i,\ P_* = K(m_* n_*)^\gamma. \tag{1}$$

Here $\Omega_* = (GM_c/r_*^3)^{1/2}$ is the Keplerian angular velocity of the fluid at the characteristic radius r_* that belongs to the Keplerian portion of the disc ; G is the gravitational constant; M_c is the total mass of the central object; Φ_* is the characteristic value of the gravitational potential; m_* and n_* are the characteristic mass and number density; m_i and n_i are the ion mass and number density; K is the dimensional constant in the polytropic law $P = Kn^\gamma$, γ is the polytrophic coefficient, ($\gamma = 5/3$ in the adiabatic case). The resulting dimensionless dynamical equations, expressed in standard cylindrical coordinates $\{r, \phi, z\}$ are:

$$\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\phi}{r}\frac{\partial V_r}{\partial \phi} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\phi^2}{r} = -\frac{1}{M_S^2}\frac{1}{n}\frac{\partial P}{\partial r} - \frac{\partial \Phi(r,z)}{\partial r}, \tag{2}$$

$$\frac{\partial V_\phi}{\partial t} + V_r\frac{\partial V_\phi}{\partial r} + \frac{V_\phi}{r}\frac{\partial V_\phi}{\partial \phi} + V_z\frac{\partial V_\phi}{\partial z} + \frac{V_r V_\phi}{r} = -\frac{1}{M_S^2}\frac{1}{nr}\frac{\partial P}{\partial \phi}, \tag{3}$$

$$\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\phi}{r}\frac{\partial V_z}{\partial \phi} + V_z\frac{\partial V_z}{\partial z} = -\frac{1}{M_S^2}\frac{1}{n}\frac{\partial P}{\partial z} - \frac{\partial \Phi(r,z)}{\partial z}, \tag{4}$$

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial rnV_r}{\partial r}+\frac{1}{r}\frac{\partial nV_\phi}{\partial \phi}+\frac{\partial nV_z}{\partial z}=0, \tag{5}$$

$$P=n^\gamma. \tag{6}$$

Here the Mach number is defined as $M_S = V_*/C_{S*}$, the characteristic sound velocity is $C_{S*} = \sqrt{P_*/(m_* n_*}$, t is time, $\{V_r, V_\phi, V_z\}$ is the plasma velocity, P and n are the pressure and density current, and Φ is the gravitational potential due to the central object. Note that a preferred direction is tacitly defined here, namely, the positive direction of the z axis is chosen according to positive Keplerian rotation.

A common property of thin Keplerian discs is their highly compressible motion with large Mach numbers [Frank, King, & Raine (2002)]:

$$\frac{1}{M_S}=\epsilon=\frac{H_*}{r_*}\ll 1, \tag{7}$$

where the characteristic effective semi-thickness H_* of the equilibrium disc is such that the disc aspect ratio ϵ equals the inverse Mach number (for the magnetic-field-free systems under consideration, the valueH_* should be fixed and known). The smallness of ϵ means that dimensionless axial coordinate is also small, i.e. $z/r_* \sim \epsilon$ ($\mid z \mid \leqslant H_*$), and consequently it is convenient to introduce the following rescaled values in order to further apply the asymptotic expansions in ϵ [see Regev (1983), Ogilvie (1997), Kluzniak & Kita (2000), Regev & Gitelman (2002), Shtemler et al. (2007), (2009)]:

$$\zeta=\frac{z}{\epsilon}\sim\epsilon^0,\ \ H_{\pm1}=\pm\frac{H_*}{\epsilon}\sim\epsilon^0,\ \ V_{z1}=\frac{V_z}{\epsilon}\sim\epsilon^0, \tag{8}$$

where the vertical velocity is scaled along with the disc height in ϵ as follows from the principle of the least degeneracy of the problem for small ϵ.

The set of Eqs. (2)-(6) is complemented with the conventional dynamic and kinematical boundary conditions on the interfaces, according to which the disc edges $z = H_\pm(t, r, \phi, z)$ are determined self-consistently. However, to simplify the problem, it is assumed that the disc edges are given by $z = \pm H_* \equiv \pm const$, where H_* the characteristic effective semi-thickness of the equilibrium disc. Consequently, the kinematic boundary conditions are not needed, while the dynamic conditions $P = n^\gamma = 0$ at the interfaces are modelled by:

$$n=0\ \text{ for }\ \zeta=\pm1. \tag{9}$$

Using of the model conditions (9) instead of the true boundary conditions at the interfaces indeed leads to the loss of the corresponding interface equations. However, this does not lead to a change in the essential nature of the dynamical behavior of the disc, and the kinematic boundary conditions provide separate equations for calculating the perturbed of the top and bottom interfaces of the disc.

The gravitational potential in Eqs. (2) and (4) may be expanded now in terms of the stretched axial variable to yield:

$$\Phi(r,\zeta)=\Phi^{(0)}(r)+\epsilon^2\Phi^{(2)}(r,\zeta)+O(\epsilon^4),\ \ \Phi^{(0)}(r)=-\frac{1}{r},\ \ \Phi^{(2)}(r,\zeta)=\frac{\zeta^2}{2r^3},\ \ r>1>>\epsilon. \tag{10}$$

Representing the solution of Eqs. (2)-(6) as a sum of the equilibrium state and its perturbations

$$P=\bar P+P',\ \ n=\bar n+n',\ \ V_r=V_r',\ \ V_\phi=\bar V_\phi+V_\phi',\ \ V_{z1}=V_z', \tag{11}$$

the steady-state solution is given by

$$\bar P(r,\zeta)=\bar n^\gamma,\ \ \bar n(r,\zeta)=\Big(\frac{\gamma-1}{2\gamma}\Big)^{\frac{1}{\gamma-1}}\Big(\frac{1-\zeta^2}{r^3}\Big)^{\frac{1}{\gamma-1}},\ \ \bar V_\phi(r)=r\Omega(r),\ \ \Omega(r)=r^{-3/2}. \tag{12}$$

Substituting Eqs. (7)-(12) into (2)-(9), linearizing about the steady-state solution, and employing the relation $P' = \bar c_S^2 n'$, yield the following linear set of equations for the perturbations to leading order in ϵ:

$$\frac{\partial V_r'}{\partial t}+\Omega(r)\frac{\partial V_r'}{\partial\phi}-2\Omega V_\phi'=0, \tag{13}$$

$$\frac{\partial V_\phi'}{\partial t}+\Omega(r)\frac{\partial V_\phi'}{\partial\phi}+\frac{1}{2}\frac{\chi^2(r)}{\Omega(r)}V_r'=0, \tag{14}$$

$$\frac{\partial \bar n V_z'}{\partial t}+\Omega(r)\frac{\partial \bar n V_z'}{\partial\phi}-\frac{2-\gamma}{\gamma-1}\Omega^2(r)\frac{d\bar c_S^2}{d\zeta}n'+\Omega^2(r)\bar c_S^2\frac{\partial n'}{\partial\zeta}=0, \tag{15}$$

$$\frac{\partial n'}{\partial t}+\Omega(r)\frac{\partial n'}{\partial\phi}+\frac{\partial \bar n V_z'}{\partial z}=-\frac{1}{r}\frac{\partial r\bar n V_r'}{\partial r}-\frac{1}{r}\frac{\partial \bar n V_\phi'}{\partial\phi}, \tag{16}$$

where

$$\chi^2=2\frac{\Omega}{r}\frac{d(r^2\Omega)}{dr},\quad \bar c_S^2(\zeta)\equiv\frac{\bar C_S^2(r,\zeta)}{\Omega^2(r)}=\frac{\gamma-1}{2}(1-\zeta^2), \tag{17}$$

$\bar{C}_S = (\partial\bar{P}/\partial\bar{n})^{1/2}$ is the dimensionless equilibrium sound velocity; χ is the epicyclic frequency. For Keplerian discs the epicyclical frequency is equals to Keplerian frequency $\chi = \Omega(r)$.

Following Eq. (7), Eqs. (13)-(16) are supplemented with the following boundary conditions:

$$n' = 0 \ \text{for} \ \zeta = \pm 1. \tag{18}$$

3 THE MODES OF WAVE PROPAGATION

3.1 Planar inertia-Coriolis waves

We start the solution of the linearized system by noticing that Eqs. (13) and (14) for V_r' and V_ϕ' are decoupled from the rest of the system due to the thin disc approximation. Furthermore, the sub-set (13)-(14) represents a set of ordinary differential equations in time and only in the spatial variable ϕ, while both the axial as well as the radial coordinates are parameters. It should be stressed that this is not an extra approximation but rather follows from the special geometry under investigation. In particular, it is noticed that that sub-set does not contain any pressure effects. The absence of pressure gradients in the horizontal dynamics hints also to the absence of the SRI in thin discs. Thus, assuming the following form for the solution of the sub-set (13)-(14):

$$\{v_r', v_\phi'\} = \exp(i\omega t - im\phi)\{\hat{v}_r, \hat{v}_\phi\} + c.c., \qquad \Big(v_r' = \frac{\bar{n}V_r'}{\Omega}, v_\phi' = \frac{\bar{n}V_\phi'}{\Omega}\Big), \tag{19}$$

where $m = 0, \pm1, \pm2, \ldots$ is the azimuthal wave number, ω is the frequency of perturbations. The reason for transforming to the scaled mass flux variables defined in Eqs. (19) is that, as will be shown below, most of the resulting equations do not depend on the radial coordinate. The corresponding dispersion relation is given by:

$$\omega = m\Omega(r) \pm \chi(r). \tag{20}$$

This represents, not surprisingly, the stable epicyclic oscillations in the disc plane. As r and ζ are mere parameters each ring $r \equiv const$, $\zeta \equiv const$ vibrates independently in its own plane. If viscous stresses are taken into account, an axial profile is imposed due to the mutual shear stresses between the rings and each entire cylindrical shell vibrates independently. This is indeed the case that has been solved in [Umurhan et al. (2006) and Rebusco et al. (2009)]. Below $\chi = \Omega(r)$ for the Keplerian rotation will be used for simplicity.

As the axial and radial coordinates play the role of passive parameters, the eigenfunctions of the inertia-Coriolis modes are determined up to arbitrary amplitude $A(r,\zeta)$:

$$\hat{v}_r(r,\zeta) = A(r,\zeta), \hat{v}_\phi(r,\zeta) = \pm i\frac{1}{2}A(r,\zeta). \tag{21}$$

As an example the following form of the amplitudes is considered

$$A(r,\zeta) = F(r)G(\zeta), \tag{22}$$

where the $F(r)$ and $G(\zeta)$ are arbitrary functions. Any special form of functions $A(r,\zeta)$, or $F(r)$ and $G(\zeta)$ represent some given set of initial conditions. That special form of the amplitude may be utilized to satisfy various regularity conditions of the solutions near the disc's edges. The regularity problem will be addressed in more details in sections 4 and 5.

Finally, it is of interest to follow the evolution in time of the radial dependence of the perturbations. Thus, defining a generalized radial wave number by, say, $k_r = i\partial lnV_r'/\partial r$, it is readily seen that it depends on time as $k_r = k_r^0 - t\lambda d\Omega/dr$ with the constant λ. Thus, even initial radially uniform perturbation will eventually develop strong radial dependence due to the rotation shear [see also Goldreich & Lynden-Bell (1965), and Balbus & Hawley (1991)]. As a result, in a general case an initial value problem has to be solved rather than a spectral one. In the current supersonically rotating thin disc case, however, due to the absence of radial partial derivative from the linearized sub-system (which is due to the insignificant role played by the radial steady-state as well as perturbed pressure gradients) a spectral problem may be solved in which the generalized time dependent radial wave number may be inferred from the solution.

3.2 Vertical sound waves

For frequencies that are different from $\omega = m\Omega(r) \pm \chi(r)$ in (20) both perturbed components of the horizontal mass flux are zero, i.e. $\hat{v}_r = 0$ and $\hat{v}_\phi = 0$. As a result, Eqs. (15) and (16) describe the perturbations in the axial velocity and the number (or mass) density. Since $\hat{v}_r = 0$ the radial derivative is once again dropped from the problem (from Eq. (16)). Thus, the following normal-wave solution is assumed:

$$\{v_z', n'\} = \exp(i\omega t - im\phi)\{\hat{v}_z, \hat{n}\} + c.c., \qquad \Big(v_z' = \frac{\bar{n}V_z'}{\Omega}\Big). \tag{23}$$

Writing $\omega = \lambda\Omega(r)$ where λ is a constant (an assumption that will be justified a posteriori) and substituting (21) - (23) into (15) -(16) yields:

$$i(\lambda - m)\hat{v}_z - \frac{2-\gamma}{\gamma-1}\frac{d\bar{c}_S^2}{d\zeta}\hat{n} + \bar{c}_S^2(\zeta)\frac{\partial \hat{n}}{\partial \zeta} = 0, \tag{24}$$

$$i(\lambda - m)\hat{n} + \frac{\partial \hat{v}_z}{\partial \zeta} = 0. \tag{25}$$

Finally, Eqs. (24) and (25) result in the following second order ordinary differential equation:

$$(1-\zeta^2)\frac{\partial^2 \hat{n}}{\partial \zeta^2} - \frac{2\gamma-3}{\gamma-1}2\zeta\frac{\partial \hat{n}}{\partial \zeta} + \alpha^2\hat{n} = 0, \quad \alpha^2 = \frac{2}{\gamma-1}[(\lambda-m)^2 + 2 - \gamma]. \tag{26}$$

In general, the solutions of Eq. (26), complemented by the boundary conditions $\hat{n} = 0$ at the disc edges $\zeta = \pm 1$, may be expressed in terms of hypergeometric functions. However, for $\gamma = 5/3$ the solutions become remarkably simple and may be expressed in terms of the following symmetric and anti-symmetric eigenfunctions (Appendix A):

$$\alpha_k = 2k+1, \quad \hat{n} = N_{2k+1}\cos(2k+1)\theta, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{27}$$

$$\alpha_k = 2k, \quad \hat{n} = N_{2k}\sin 2k\theta, \quad k = \pm 1, \pm 2, \ldots, \tag{28}$$

where $\theta = \arcsin\zeta$. Thus, according to (26) the following dispersion relation is obtained for $\gamma = 5/3$:

$$\lambda = \lambda_{\pm}^{(m,k)} = m \pm \sqrt{\frac{\alpha_k^2 - 1}{3}}, \quad m = 0, \pm 1, \pm 2, \ldots. \tag{29}$$

The admissible values of α_k and k are given by either (27) or (28). These eigenvalues describe stable stratified sound waves that propagate independently on the surfaces of each individual cylindrical shell. In a co-rotating frame those are standing sound waves whose dependence on the axial coordinate is described by Eqs. (27) or (28), where k plays the role of an axial wave number. Thus, according to (27) or (28) as k is increased the waves concentrate in growing numbers towards the disc's edges. It is finally emphasized that the compressibility effects in the vertical direction are indeed essential for the thin disc approximation as the sound vertical crossing time is of the order of Ω^{-1}.

The seemingly simple dispersion relations (20) and (29) actually entail an important conclusion. As they exhaust all possible eigenvalues of the linearized system it means that in the limit of thin disc approximation, the disc is spectrally stable against hydrodynamic perturbations. Thus, the asymptotic in time behavior of the linearized system exhibit two stable families of normal mode oscillations within the thin discs; one consists of vibrating rings due to inertia-coriolis effects, while the other one imposes density fluctuation on each individual cylindrical shell in the form of standing sound waves. In particular, it is obvious that no unstable strato-rotational modes may be excited in such systems. This is due to the absence of radial and azimuthal perturbed pressure gradients from Eqs. (13) and (14), which leads to the decoupling of the vertical density (strato) from the planar (rotational) dynamics. In contrast to that picture, in Couette flows, such pressure gradients play an important role in coupling of the planar and vertical perturbations and thus giving rise to the strato-rotational instability. The reason for the importance of the planar pressure gradients in Couette flows is the presence of a rigid wall, which is absent in rotating discs in thin disc approximation.

Having clarified the time asymptotic (marginal) stability of rotating discs, this is by no means the end of the hydrodynamical story. Short time dynamics may exhibit significant amplification of initial small perturbations. As will be shown below, two mechanisms are responsible for such behavior, namely resonant as well as non-resonant excitation of vertical sound waves by planar inertia-coriolis modes. Furthermore, the algebraically growing perturbations can create the conditions for the re-appearing of the SRIs. It should be mentioned that Umurhan et al. (2006) and Rebusco et al. (2009) have described the non resonant mechanism in discs that are characterized by a phenomenological viscosity, by expansion in a single small parameter. The current work explores asymptotic expansions in the small but finite aspect ratio of the steady-state disc, ϵ, for both equilibrium and perturbations, while the amplitude of the perturbations is assumed to be infinitesimal small independent value. The resonant mechanism is described here for the first time.

4 NON-RESONANTLY DRIVEN SOUND WAVES

Returning to Eqs. (15) and (16) that describe the dynamics of the vertical sound waves, it is clear that the latter may be excited by the inertia-coriolis modes that are eigen functions of the decoupled system (13)-(14). Moreover, due to the rotation shear, the amplitude of the driven sound modes grows linearly in time. This is illustrated by Eq. (16) which may be cast with the aid of expressions (19)-(22) in the following form of the auxiliary inhomogeneous equation:

$$\frac{1}{\Omega}\frac{\partial n'}{\partial t} + \frac{\partial n'}{\partial \phi} + \frac{\partial v_z'}{\partial z} = -\Big(it\lambda\frac{d\Omega}{dr} + \frac{1}{F}\frac{dF}{dr} + \frac{1}{r\Omega}\frac{dr\Omega}{dr}\Big)v_r' + i\frac{m}{r}v_\phi'. \tag{30}$$

The solutions of the inhomogeneous Eqs. (15)-(16) are represented now in the following way:

$$\{v_r', v_\phi', v_z', n'\} = \exp(i\omega t - im\phi)\{\hat{v}_r, \hat{v}_\phi, \hat{v}_z, \hat{n}\} + c.c. \tag{31}$$

Here $\omega = \lambda\Omega(r)$ ($\lambda \equiv const$) represents the eigen-frequency of the inertia-coriolis waves, i.e., $\lambda = m \pm 1$, $(m = 0, \pm 1, \pm 2, \ldots)$ according to Eq. (20). In addition it is assumed that

$$\hat{v}_r = \hat{v}_r^{(0)}, \hat{v}_\phi = \hat{v}_\phi^{(0)}, \hat{v}_z = \hat{v}_z^{(0)} + it\Omega\hat{v}_z^{(1)}, \hat{n} = \hat{n}^{(0)} + it\Omega\hat{n}^{(1)}. \tag{32}$$

Here the superscript denotes the order in time; $\hat{v}_r^{(0)}$, $\hat{v}_\phi^{(0)}$ are given by Eqs. (21),(22) while relations for $\hat{n}^{(k)}$ and $\hat{v}_z^{(k)}$ $(k = 0, 1)$ are derived after some algebra by substituting (31) and (32) into (15) - (16) and equating equal powers of t:

$$\pm i\hat{n}^{(1)} + \frac{\partial \hat{v}_z^{(1)}}{\partial\zeta} = -\lambda\frac{1}{\Omega}\frac{d\Omega}{dr}\hat{v}_r, \tag{33}$$

$$\pm i\hat{v}_z^{(1)} - \frac{2-\gamma}{\gamma-1}\frac{d\bar{c}_S^2}{d\zeta}\hat{n}^{(1)} + \bar{c}_S^2(\zeta)\frac{\partial\hat{n}^{(1)}}{\partial\zeta} = 0, \tag{34}$$

and

$$\pm i\hat{n}^{(0)} + \frac{\partial \hat{v}_z^{(0)}}{\partial\zeta} = -i\hat{n}^{(1)} - \frac{1}{r\Omega}\frac{\partial r\Omega\hat{v}_r}{\partial r} + \frac{im}{r}\hat{v}_\phi, \tag{35}$$

$$\pm i\hat{v}_z^{(0)} - \frac{2-\gamma}{\gamma-1}\frac{d\bar{c}_S^2}{d\zeta}\hat{n}^{(0)} + \bar{c}_S^2(\zeta)\frac{\partial\hat{n}^{(0)}}{\partial\zeta} = -i\hat{v}_z^{(1)}, \tag{36}$$

with the following boundary conditions:

$$\hat{n}^{(0)} = \hat{n}^{(1)} = 0 \text{ for } \zeta = \pm 1. \tag{37}$$

As is indeed seen from Eqs. (33) - (36) the sound waves are driven by the inertial mode, while the linear growth in time is entirely due to effect of the rotation shear. Substituting now Eqs. (21), (22) into Eqs. (33), (34) and (35), (36) yields the following two decoupled subsystems:

$$(1-\zeta^2)\frac{\partial^2\hat{n}^{(1)}}{\partial\zeta^2} - \frac{2\gamma-3}{\gamma-1}2\zeta\frac{\partial\hat{n}^{(1)}}{\partial\zeta} + 2\frac{3-\gamma}{\gamma-1}\hat{n}^{(1)} = \pm 2i\frac{m\pm1}{\gamma-1}\frac{F(r)}{\Omega(r)}\frac{d\Omega}{dr}G(\zeta), \tag{38}$$

$$\hat{v}_z^{(1)} = \pm i(2-\gamma)\zeta\hat{n}^{(1)} \pm i\frac{\gamma-1}{2}(1-\zeta^2)\frac{\partial\hat{n}^{(1)}}{\partial\zeta}, \tag{39}$$

and

$$(1-\zeta^2)\frac{\partial^2\hat{n}^{(0)}}{\partial\zeta^2} - \frac{2\gamma-3}{\gamma-1}2\zeta\frac{\partial\hat{n}^{(0)}}{\partial\zeta} + 2\frac{3-\gamma}{\gamma-1}\hat{n}^{(0)} = \mp\frac{4}{\gamma-1}\hat{n}^{(1)} + i\frac{1}{\gamma-1}\Big[2(m\pm1)\frac{F}{\Omega}\frac{d\Omega}{dr} + im\frac{F}{r} \pm \frac{2}{r\Omega}\frac{drF\Omega}{dr}\Big]G, \tag{40}$$

$$\hat{v}_z^{(0)} = \pm i(2-\gamma)\zeta\hat{n}^{(0)} \pm i\frac{\gamma-1}{2}(1-\zeta^2)\frac{\partial\hat{n}^{(0)}}{\partial\zeta} \mp \hat{v}_z^{(1)}. \tag{41}$$

Equations (38) and (40), subject to the boundary conditions (37) and either symmetric or anti-symmetric scaling function $G(\zeta)$ are consistent with either symmetric or anti-symmetric solutions $\hat{n}^{(k)}(\zeta)$ $(k = 0, 1)$, i.e:

$$\hat{n}^{(k)}(\zeta) = \hat{n}^{(k)}(-\zeta), \tag{42}$$

$$\hat{n}^{(k)}(\zeta) = -\hat{n}^{(k)}(-\zeta). \tag{43}$$

A particular family of solutions is considered which is characterized by the following form of the free functions $F(r)$ and $G(\zeta)$ for $\gamma = 5/3$ (Appendix A):

$$F(r) = r, \quad G(\zeta) = G_3\cos^3\theta + G_5\cos^5\theta + G_7\cos^7\theta, \quad (\theta = \arcsin\zeta). \tag{44}$$

As is shown in Appendix A3, such choice of $F(r)$ and $G(\zeta)$ results in radial independence of the perturbed number density and its regularity at the disc edges. In that case the problem is reduced to two second order ordinary differential equations for $\hat{n}^{(0)}$ and $\hat{n}^{(1)}$ complemented by the corresponding boundary conditions (42) or (43) at the disc edges, while $\hat{v}_z^{(0)}$ and $\hat{v}_z^{(1)}$ may be calculated from (39) and (41). Thus, using (44) the following symmetric solutions for $\hat{n}^{(0)}$ and $\hat{n}^{(1)}$ are obtained:

$$\hat{n}^{(k)}(\zeta) = N_3^{(k)}\cos^3\theta + N_5^{(k)}\cos^5\theta + N_7^{(k)}\cos^7\theta, \tag{45}$$

where the coefficients $N_j^{(k)}$ and G_j $(j = 3, 5, 7,\ k = 0, 1)$ are independent of the radius. Thus, for each value m there are two linearly growing modes except for $m = \pm 1$, where for each m corresponds only a single growing mode. It is finally noted that by the special construction of the solution in Appendix A3 both right hand sides of Eqs. (38) and (40) satisfy the solvability condition, namely that they are orthogonal to the solution of the common homogeneous part.

The special choice of the free function $G(\zeta)$ presented in Eq. (44) indeed assures the regularity of all components of the velocity vector $\mathbf{V}'$. However, such regularity requirements are rather too stringent and actually the sharp boundary model used in the current work does allow certain degree of singularity in the velocity. Thus, more relaxed conditions that are easily satisfied are that the perturbed components of the mass flux (or the momentum per unit volume), i.e. $\bar{n}\mathbf{V}'$ is zero at the disc's edges. Indeed, such singularity in the velocity components is a direct consequence of the sharp boundary disc with zero equilibrium-density at the horizontal disc edges ($\zeta = \pm 1$) that has been employed in the present investigation. In fact, astrophysical discs have no well defined boundaries, the disc boundary is not sharp, and the vacuum boundary conditions imposed at a finite height are rather a good approximation than an exact model. According to a more realistic model that assumes a diffused disc with exponentially decreasing density [see e.g. Balbus & Hawley (1991) for thin isothermal discs], the density has a small but finite value at the effective boundary of the disc. With that in mind the singularity at the disc edges may be regularized within a model with a small but non zero pressure value at the effective disc edges. The regularized model yields large but finite values of the fluid velocities at the disc edge that demonstrate that non modal growth may indeed leads to significant amplification of the perturbations.

An example of a singular case is:

$$F(r) = r, \quad G(\zeta) = G_1 \cos\theta, \quad (\theta = \arcsin\zeta). \tag{46}$$

Again such choice of $F(r)$ provides the solutions for $\hat{n}^{(k)}$ $(k = 0, 1)$ which are independent of the radius:

$$\hat{n}^{(k)} = N_1^{(k)} \cos\theta, \quad (N_1^{(1)} = \mp i\frac{3}{2}(m \pm 1)G_1, \quad N_1^{(0)} = -i(4m \pm 5)G_1). \tag{47}$$

In this case the velocity indeed has singularity at the disc's edges, however due to the more rapid decrease of the steady state number density the mass flux is zero.

5 RESONANTLY DRIVEN SOUND WAVES

As can be seen from Eq. (30), stable sound modes may be excited resonantly by the inertial modes if any pair of respective eigen-values (20) (with $\chi = \Omega(r)$, $\omega = \lambda_{\pm}^{(m)}\Omega(r)$, where $\lambda_{\pm}^{(m)} = m\pm 1$) and (29) are equal, i.e. $\lambda_{\pm}^{(m)} = \lambda_{\pm}^{(m,k)}$ $(m = 0, \pm 1, \pm 2, \ldots,$ $k = 0, 1)$. It is easy to show that this happens only for the first anti-symmetric mode (28) for arbitrary m $(\gamma = 5/3)$ and:

$$k = \pm 1, \quad \alpha_k = \pm 2, \quad \lambda_{\pm}^{(m,k)} = m \pm 1, \quad \hat{n} = \pm N_2 \sin 2\theta, \quad (\theta = \arcsin\zeta). \tag{48}$$

The solution of Eq. (30) then, that describes the resonant coupling between sound and inertial waves in Keplerian discs is thus represented by:

$$\{v_r', v_\phi', v_z', n'\} = \exp(i\omega t - im\phi)\{\hat{v}_r, \hat{v}_\phi, \hat{v}_z, \hat{n}\} + c.c., \quad \omega = \lambda\Omega(r), \quad (\lambda \equiv const). \tag{49}$$

The combination of resonant coupling with the effect of the rotation shear leads to the following form of the solution:

$$\hat{v}_r = \hat{v}_r^{(0)}, \; \hat{v}_\phi = \hat{v}_\phi^{(0)}, \; \hat{v}_z = \hat{v}_z^{(0)} + it\Omega\hat{v}_z^{(1)} + \frac{(it\Omega)^2}{2}\hat{v}_z^{(2)}, \; \hat{n} = \hat{n}^{(0)} + it\Omega\hat{n}^{(1)} + \frac{(it\Omega)^2}{2}\hat{n}^{(2)}, \tag{50}$$

where the superscripts denote the order in time; $\hat{v}_r^{(0)}$, $\hat{v}_\phi^{(0)}$ are given by (21), (22) while $\hat{v}_z^{(k)}$, $\hat{n}^{(k)}$, $(k = 0, 1, 2)$ are determined by the relations below. It should be noted that resonant interaction between the two modes of wave propagation may occur also in a shearless rotation. In that case the amplitude of the sound waves grows only linearly in time. It is the combined effect of resonant interaction and rotation shear that gives rise to a quadratic growth in time that is described in Eq. (50). Substituting (49)- (50) into (13)) - (16) and equating equal powers of t yields

$$\pm i\hat{n}^{(2)} + \frac{\partial\hat{v}_z^{(2)}}{\partial\zeta} = 0, \tag{51}$$

$$\pm i\hat{v}_z^{(2)} - \frac{2-\gamma}{\gamma-1}\frac{d\bar{c}_S^2}{d\zeta}\hat{n}^{(2)} + \bar{c}_S^2(\zeta)\frac{\partial\hat{n}^{(2)}}{\partial\zeta} = 0, \tag{52}$$

and

$$\pm i\hat{n}^{(1)} + \frac{\partial\hat{v}_z^{(1)}}{\partial\zeta} = -i\hat{n}^{(2)} - \lambda\frac{1}{\Omega}\frac{d\Omega}{dr}\hat{v}_r \tag{53}$$

$$\pm i\hat{v}_z^{(1)} - \frac{2-\gamma}{\gamma-1}\frac{d\bar{c}_S^2}{d\zeta}\hat{n}^{(1)} + \bar{c}_S^2(\zeta)\frac{\partial\hat{n}^{(1)}}{\partial\zeta} = 0, \tag{54}$$

and

$$\pm i\hat{n}^{(0)} + \frac{\partial\hat{v}_z^{(0)}}{\partial\zeta} = -i\hat{n}^{(1)} - \frac{1}{r\Omega}\frac{\partial r\Omega\hat{v}_r}{\partial r} + \frac{im}{r}\hat{v}_\phi, \tag{55}$$

$$\pm i\hat{v}_z^{(0)} - \frac{2-\gamma}{\gamma-1}\frac{d\bar{c}_S^2}{d\zeta}\hat{n}^{(0)} + \bar{c}_S^2(\zeta)\frac{\partial \hat{n}^{(0)}}{\partial \zeta} = -i\hat{v}_z^{(1)}, \tag{56}$$

while the planar dynamics is described by Eqs. (21), (22) for $\hat{v}_r$ and $\hat{v}_\phi$. The number densities $\hat{n}^{(k)}$ $(k = 0, 1, 2)$ are supplemented by the boundary conditions:

$$\hat{n}^{(0)} = \hat{n}^{(1)} = \hat{n}^{(2)} = 0 \;\text{ for }\; \zeta = \pm 1. \tag{57}$$

Equations (55) and (56) for $\hat{n}^{(2)}$ and $\hat{v}_z^{(2)}$ form a homogeneous sub-system for the sound mode:

$$(1-\zeta^2)\frac{\partial^2 \hat{n}^{(2)}}{\partial \zeta^2} - \frac{2\gamma-3}{\gamma-1}2\zeta\frac{\partial \hat{n}^{(2)}}{\partial \zeta} + 2\frac{3-\gamma}{\gamma-1}\hat{n}^{(2)} = 0, \tag{58}$$

$$\hat{v}_z^{(2)} = \pm i(2-\gamma)\zeta\hat{n}^{(2)} \pm i\frac{\gamma-1}{2}(1-\zeta^2)\frac{\partial \hat{n}^{(2)}}{\partial \zeta}. \tag{59}$$

The only solutions of Eq. (58) that satisfy the resonance conditions and which are zero at $\zeta = \pm 1$ and satisfy (57) are given by ($\gamma = 5/3$):

$$\hat{n}^{(2)} = \pm N_2^{(2)} \sin 2\theta, \;\; \hat{v}_z^{(2)} = iN_2^{(2)}[(2-\gamma)\sin\theta\sin 2\theta + (\gamma-1)\cos\theta\cos 2\theta], \;\; (\theta = \arcsin\zeta). \tag{60}$$

In general, $N_2^{(2)}$ may depend on the radial variable, however, as an example it will be considered as a constant. Also, it is noted that in contrast to the non-resonant case, here singularity in the velocity is inevitable. However, as stated above, the less restrictive condition is satisfied as the mass flux is regular and vanishes at the disc's edges.

Moving on the following equations for $\hat{n}^{(1)}$, $\hat{v}_z^{(1)}$ and $\hat{n}^{(0)}$, $\hat{v}_z^{(0)}$:

$$(1-\zeta^2)\frac{d^2 \hat{n}^{(1)}}{d\zeta^2} - \frac{2\gamma-3}{\gamma-1}2\zeta\frac{d\hat{n}^{(1)}}{d\zeta} + 2\frac{3-\gamma}{\gamma-1}\hat{n}^{(1)} = \mp\frac{2}{\gamma-1}\hat{n}^{(2)} \mp 3i\frac{m\pm 1}{\gamma-1}G(\zeta), \tag{61}$$

$$\hat{v}_z^{(1)} = \pm i(2-\gamma)\zeta\hat{n}^{(1)} \pm i\frac{\gamma-1}{2}(1-\zeta^2)\frac{d\hat{n}^{(1)}}{d\zeta}, \tag{62}$$

and

$$(1-\zeta^2)\frac{d^2 \hat{n}^{(0)}}{d\zeta^2} - \frac{2\gamma-3}{\gamma-1}2\zeta\frac{d\hat{n}^{(0)}}{d\zeta} + 2\frac{3-\gamma}{\gamma-1}\hat{n}^{(0)} = -\frac{2}{\gamma-1}\hat{n}^{(2)} \mp \frac{4}{\gamma-1}\hat{n}^{(1)} - 2i\frac{m\pm 1}{\gamma-1}G, \tag{63}$$

$$\hat{v}_z^{(0)} = \pm i(2-\gamma)\zeta\hat{n}^{(0)} \pm i\frac{\gamma-1}{2}(1-\zeta^2)\frac{d\hat{n}^{(0)}}{d\zeta} \mp \hat{v}_z^{(1)}. \tag{64}$$

The homogeneous part of Eq. (61) complemented by corresponding boundary conditions for $\hat{n}^{(1)}$ coincides with the above eigenvalue problem for $\hat{n}^{(2)}$. Hence non-trivial solutions of the inhomogeneous equations exist if the right hand sides of (61) satisfies the solvability condition, namely, that it is orthogonal to $\hat{n}_2$ as it is given in (60). This means that in order to obtain resonant coupling, the vertical dependence of the inertia-coriolis waves must contain a term of the following shape:

$$G(\zeta) = G_2 \sin 2\theta, \;\; (\theta = \arcsin\zeta), \tag{65}$$

where G_2 is an arbitrary amplitude. For simplicity it will be assumed that Eq. (65) indeed describes the entire profile of the driving waves and in addition that $F(r) = r$. As will be seen later, the latter results in simple radial independent solutions. Imposing now the solvability condition, $N_2^{(2)}$ is given by:

$$N_2^{(2)} = \mp i\frac{3}{2}(m\pm 1)G_2. \tag{66}$$

Eq. (61) for $\hat{n}^{(1)}$ may be solved now. Due to imposing the solvability condition and the special form of G given in Eq. (65), the right hand side of that equation is zero. Hence its solution is given by $N_2^{(1)}\sin 2\theta$. Turning now to Eq. (63) it is immediately seen that the same solvability condition determines now $N_2^{(1)}$ to be:

$$N_2^{(1)} = \pm i\frac{1}{4}(m\mp 1)G_2. \tag{67}$$

Summarizing, the solutions for $\hat{n}^{(1)}$, $\hat{v}_z^{(1)}$ and $\hat{n}^{(0)}$, $\hat{v}_z^{(0)}$ are ($\theta = \arcsin\zeta$):

$$\hat{n}^{(1)} = N_2^{(1)}\sin 2\theta, \;\; \hat{v}_z^{(1)} = \pm iN_2^{(1)}[(2-\gamma)\sin\theta\sin 2\theta + (\gamma-1)\cos\theta\cos 2\theta], \tag{68}$$

$$\hat{n}^{(0)} = N_2^{(0)}\sin 2\theta, \;\; \hat{v}_z^{(0)} = i[\pm N_2^{(0)} - N_2^{(1)}\Omega(r)][(2-\gamma)\sin\theta\sin 2\theta + (\gamma-1)\cos\theta\cos 2\theta]. \tag{69}$$

In the above solution (66) - (69) there are two arbitrary constants G_2 and $N_2^{(0)}$. The first one represents the arbitrary amplitude of the driving inertia-coriolis waves while the second one describes free sound oscillations. Finally, it is noticed again that while the vertical component of the perturbed velocity is singular at the disc's edges, the corresponding component of the mass flux tends to zero.

6 CONCLUSIONS

A comprehensive study of the long as well as short time hydrodynamic response of Keplerian rotating discs to small perturbations has been carried out for realistic thin disc geometry. Explicit analytical solutions of the classical eigenvalue problem reveal that thin discs are hydrodynamically marginally stable asymptotically in time. Such a picture naturally leads to the ruling out of the strato-rotational instabilities as primary contributors to the dynamical evolution of thin ultrasonically rotating discs. This has been shown to be a direct result of the decoupling of the planar dynamics from the vertical acoustics, which in turn is a result of the negligible role played by radial pressure gradients. The latter is indeed the agent of coupling of the strato and rotational modes that lead to instability in the case of Couette flows due to the presence of solid radial boundaries. Notwithstanding the spectral stability of thin rotating discs, they have been shown to exhibit intense hydrodynamical activity in the form of algebraic short-term growth of initially small perturbations. Two mechanisms have been identified and studied that lead to such non modal dynamical behavior. The first one is the non resonant excitation of co-rotating standing sound waves on individual cylindrical shells by planar inertia-coriolis modes. Such coupling between the modes occurs due to the rotation shear and results in a linear in time growth of the perturbations. The second mechanism is the resonant driving of the sound waves by the inertia-coriolis modes. While non resonant coupling relies on the non normal nature of the linearized system of the dynamical equations, resonant coupling exists also for a shearless rotation. However, in the presence of rotation shear the perturbations grow quadratically in time. Both mechanisms are powered by the compressible motion of the fluid. Thus, compressible inertia-coriolis stable oscillations continuously pump energy from the sheared steady state flow and transfer it (resonantly as well as non resonantly) to the continuously algebraically growing sound waves. Compressibility is indeed inevitable due to the supersonic rotation of the steady-state disc combined with its small vertical dimensions which make the vertical sound crossing time of the order of a rotation period. Simple explicit analytical solutions were obtained for all possible cases. It should be stressed that non modal growth has been exhibited for all admissible parameters of the system.

Along side with the non modal growth of small perturbations, the rotation shear is also responsible for the linear growth in time of an effective radial wave number. Such a process naturally gives rise to enhanced perturbed radial pressure gradients. In light of the discussion of the special role played by the latter in the onset of the SRI, it may be speculated, that such a development may reinstate the SRI as an important dynamical process due to secondary instabilities that occur in the wake of the primary non modal hydrodynamical growth. Thus, the linear as well as quadratic amplification of initial small perturbation as well as the secondary instabilities they entail could provide a viable mechanism for sustaining turbulence and consequently to enhanced transport coefficients.

A detailed description of nonmodal growth has been presented before in Umurhan et al. (2006) and Rebusco et al. (2009). The main differences between those works and the current investigation are:

- In Umurhan et al. (2006) and Rebusco et al. (2009) a finite amplitude non-linear solution to the problem of the dynamical evolution of the full equations was sought. In that approach it is assumed that solutions of the equations, both steady as well as dynamical, may be expanded in a single small parameter, namely, ϵ. Solutions are then determined by iteratively solving the resulting equations at each order of ϵ. Consequently, the forcing terms at each order are the solutions of the previous order which results in algebraic growth. In contrast, in the current work a classical linearization procedure is employed in which an extra independent small parameter is introduced that measures the amplitude of the initial perturbations. It is due to the negligible role played by the radial pressure gradients that the resulting linear system of equations is decoupled into two subsystems, one of which describes the perturbed planar velocities while the other one determines the dynamical evolution of the vertical sound waves. Moreover, the planar modes act as forcing terms for the vertical sound waves. This mechanism gives rise to the resonant as well as the nonresonant algebraic growth of the latter.
- Umurhan et al. (2006) and Rebusco et al. (2009) have demonstrated the occurrence of linear growth (due to the shear of the steady state rotation) that results from the nonresonant driving of the vertical sound waves by the planar dynamics. In the current work, in addition to that linear growth, a resonant process is identified which results in a quadratic growth in time. Such second order growth is a result of the combined effect of the resonance between the vertical and and the planar modes and the shear of the steady state rotation. Furthermore, the resonance is a global one in the sense that it occurs all over the disk; the planar vibrations of a ring of any radius resonantly rattle the cylindrical shell where it resides.
- Unlike the inviscid limit that is investigated in the current work, Umurhan et al. (2006) and Rebusco et al. (2009) present the solution of the viscous problem where the viscous stresses are modelled by the Shakura-Sunyaev α parameter. Umurhan et al. (2006) and Rebusco et al. (2009) refer to values around $\alpha = 10^{-3}$ as representing "realistic" values as they are close to the result found numerically in subcritical hydrodynamic transitions in Lesur & Longaretti (2005). Numerical solutions indicate that for such small values of the viscosity (or large values of the Reynold number), the algebraic growth proceeds undisturbed at least for 10^3 rotation times, during which the amplitude grows by a factor of 10^4, before viscous damping starts to play any significant role. Thus, the dynamical evolution during the first non-viscous long period of time is independent of the viscosity and is well described by the inviscid limit. Indeed, the eigenvalues of the planar dynamics [Eq. (63) in Rebusco et al. (2009)] as well as the temporal behavior of the driven sound waves [Eq. (66) in Rebusco et al. (2009)] transit to Eq. (20) and Eqs. (31) Eqs. (32), respectively of the current work in the limit $\alpha \to 0$. The singular limit of zero viscosity is revealed then only in

the spatial profiles of the eigenfunctions. Thus, the spatial profiles obtained in the current work may be considered as outer solutions of a boundary layer problem which are valid in most parts of the disk except perhaps within a thin region next to the vertical edges of the disk. The advantage of the inviscid approach however is that it enables to derive simple analytical expressions for the eigenvalues and the eigenfunctions such as those given in Eqs. (27)-(29), and consequently to identify the resonant quadratic growth. The inviscid problem has also been solved by Umurhan & Shaviv (2005) where they have described the nonresonant growth with a single parameter expansion (see first item).

Obviously, the algebraically growing perturbations (both resonant as well as non resonant) do not keep growing without bound. Thus, nonlinear effects as well as viscous damping are expected to quench the linear or quadratic growth of the perturbations. As has been demonstrated in Rebusco et al. (2009), for realistic values of the viscosity, the algebraic growth of nonresonantly driven sound waves takes place unhindered for at least 10^3 rotation times before viscous damping kicks in. During that time the initially small perturbations grow significantly by few orders of magnitude. It is expected therefore, that nonlinear redistribution of the perturbations energy among other length scales will occur before the perturbations are damped by viscous effects. In the case of resonantly driven sound waves the growth of the perturbation is even faster. In addition, as $|k_r| \sim |t\lambda d\Omega/dr|$ (see Section 3.1) the time needed to develop significant perturbed radial pressure gradients is of the order of $(\lambda\epsilon)^{-1}$. According to Rebusco et al. (2009) there is enough time for that to occur before viscous effects become important and even before nonlinear effects take over. Moreover, the time to establish significant radial gradients decreases as the azimuthal mode number is increased. Also, according to Dubrulle et al. (2005) and Shalybkov & R*ü*diger (2005), the maximal growth rates of the SRI are achieved for Froude numbers of the order of $\sim 1-10$ and are of the order of 10^{-1} inverse rotation periods. This, again, leaves ample time for the growth of secondary SRI instabilities before the perturbations are killed off by viscous stresses. It should be mentioned however that in Dubrulle et al. (2005) as well as in Shalybkov & R*ü*diger (2005) (as indeed in most SRI investigations) the Froude number has been assumed to be constant. This is very different from realistic thin disk configurations where the Froude number is given according to eqs. (10) and (12) by $Fr^2 = 3\zeta^2/(1-\zeta^2)$ and thus varies between zero and infinity over the small vertical extent of the disk. Hence, the numbers cited above are used just as a crude estimation and the true nature of a possible secondary instability may be investigated at this stage only by numerical simulations.

It should finally be stressed once again that the calculations presented above are valid in the Keplerian portion of the disk, namely far away from the radius of zero torque. In regions around the latter, which are much closer to the central object, the radial pressure gradients may play an important role in the dynamics of small perturbations, as is shown in Reynolds & Miller (2009) who studied numerically g-mode trapping that occur within 8 - 32 Schwarzschild radii from a central black hole.

ACKNOWLEDGMENTS

This work has been carried out while one of us (M.M) was a Mercator Professor at the university of Potsdam. He would like to thank the Physics department at the University of Potsdam and the Astrophysikalisches Institut Potsdam for their hospitality. The authors thank Georgy Burde for useful and insightful discussions.

APPENDIX A:

A1 The basic boundary-value problem

Equations (26), (38), (40), (58) and (61), (63) have the following general form (it was chosen $F(r) = r$ everywhere through that section):

$$(1-\zeta^2)\frac{d^2n}{d\zeta^2} - \frac{2\gamma-3}{\gamma-1}2\zeta\frac{dn}{d\zeta} + \alpha^2 n = \Psi(\zeta), \quad |\,\zeta\,| < 1, \ \ \alpha^2 = \frac{2}{\gamma-1}[(\lambda-m)^2 + 2 - \gamma]. \tag{A1}$$

Equation (A1) is supplemented by the boundary conditions at the disc edges. In addition, the solutions of Eq. (A1) satisfy the same symmetry as that of the driving term on the right hand side, namely either symmetric ($\Psi(\zeta) = \Psi(-\zeta)$) or anti-symmetry ($\Psi(\zeta) = -\Psi(-\zeta)$) conditions:

$$\frac{dn}{d\zeta} = 0 \ \text{ for } \ \zeta = 0, \ \ n = 0 \ \text{ for } \ \zeta = 1, \tag{A2}$$

$$n = 0 \ \text{ for } \ \zeta = 0, \ \ n = 0 \ \text{ for } \ \zeta = 1. \tag{A3}$$

Transforming to a new independent variable $\theta = \arcsin\zeta$, problem (A1)-(A3) assumes the following form:

$$\frac{d^2n}{d\theta^2} - \frac{3\gamma-5}{\gamma-1}\tan\theta\frac{dn}{d\theta} + \alpha^2 n = \Psi(\zeta), \ \ -\pi/2 < \theta < \pi/2, \tag{A4}$$

$$\frac{dn}{d\theta} = 0 \ \text{ for } \ \theta = 0, \ \ n = 0 \ \text{ for } \ \theta = \pi/2, \tag{A5}$$

$$n = 0 \ \text{ for } \ \theta = 0, \ \ n = 0 \ \text{ for } \ \theta = \pi/2. \tag{A6}$$

For arbitrary values of γ the solutions of Eq. (A4) may be expressed in terms of the hypergeometric functions. However, for the specific value $\gamma = 5/3$ Eq. (A4) becomes remarkably simple. Conveniently therefore, considering from now on the case with $\gamma = 5/3$ as well as Keplerian rotation, $\Omega = r^{-3/2}$.

A2 The eigenvalue problem for the sound mode

The eigenvalue problem for the sound modes is given by (see Eq. (26) for $\gamma = 5/3$):

$$\frac{d^2 n}{d\theta^2} - +\alpha^2 n = 0, \ \ 0 < \theta < \pi/2, \tag{A7}$$

$$n = 0 \ \text{ for } \ \theta = \pm\pi/2. \tag{A8}$$

The general solution of Eq. (A7) is:

$$n = C_1 \cos\alpha\theta + C_2 \sin\alpha\theta, \tag{A9}$$

where $\cos\alpha\theta$ and $\sin\alpha\theta$ are the symmetric and anti-symmetric eigenfunctions. Imposing boundary conditions (A5) and (A6), the following two families of eigenvalues and eigenfunctions obtained:

$$\alpha = 2k+1, \ \ n = N_{2k+1}\cos(2k+1)\theta, \ \ k = 0, \pm 1, \pm 2, \ldots, \tag{A10}$$

$$\alpha = 2k, \qquad n = N_{2k}\sin 2k\theta, \qquad\qquad k = \pm 1, \pm 2, \ldots. \tag{A11}$$

A3 The regular symmetric solution for the non-resonantly driven sound mode

The following choice of a symmetric driving function $\Psi(\zeta)$ in Eq. (A4) is considered (see Eqs. (38) and (40) with $\gamma = 5/3$):

$$\frac{1}{4}\frac{d^2 n}{d\theta^2} + n = \frac{1}{4}\Psi(\zeta) \equiv \psi_3 \cos^3\theta + \psi_5\cos^5\theta + \psi_7\cos^7\theta, \ \ 0 < \theta < \pi/2, \tag{A12}$$

$$\frac{dn}{d\theta} = 0 \ \text{ for } \ \theta = 0, \ \ n = 0 \ \text{ for } \ \theta = \pi/2, \tag{A13}$$

where $\lambda = m \pm 1$, $\alpha^2 = 4$. The motivation for such a choice of the right hand side in eq. (A12) is the following: recalling solutions (20) and (21) for v'_r and v'_ϕ, as well as definition (19), and the steady state solution for the number density (12) [$\bar{n} \sim (1-\zeta^2)^{3/2} \sim \cos^3\theta$], it is seen that the solution for V'_r and V'_ϕ that are regular at the disk edges $\zeta^2 = 1$ are given by a function G that contains powers of third order and higher of $\cos\theta$. It will consequently be shown in this subsection that similar regularity requirements on V'_z imply that G must contain at least also the next two higher odd powers of $\cos\theta$ whose amplitudes are determined by the arbitrary amplitude of the cubic term.

The general solution of (A12)) for symmetric modes is

$$n = C_2\cos 2\theta + C_1\cos\theta + C_3\cos 3\theta + C_5\cos 5\theta + C_7\cos 7\theta, \tag{A14}$$

where from (A13) at $\theta = \pi/2$ follows that $C_2 = 0$, and

$$C_1 = \psi_3 + \frac{5}{6}\psi_5 + \frac{35}{48}\psi_7, \ \ C_3 = -\frac{1}{5}\psi_3 - \frac{1}{4}\psi_5 - \frac{21}{80}\psi_7, \ \ C_5 = -\frac{1}{84}\psi_5 - \frac{1}{48}\psi_7, \ \ C_7 = -\frac{1}{720}\psi_7. \tag{A15}$$

For further convenience (A14) is rewritten in the form

$$n = N_1\cos\theta + N_3\cos^3\theta + N_5\cos^5\theta + N_7\cos^7\theta, \tag{A16}$$

where

$$N_1 = C_1 - 3C_3 + 5C_5 - 7C_7, \ \ N_3 = 4C_3 - 20C_5 + 56C_7, \ \ N_5 = 16C_5 - 112C_7, \ \ N_7 = 64C_7,$$

or using (A16)

$$N_1 = \frac{8}{5}\psi_3 + \frac{32}{21}\psi_5 + \frac{64}{45}\psi_7, \ \ N_3 = -\frac{4}{5}\psi_3 - \frac{16}{21}\psi_5 - \frac{32}{45}\psi_7, \ \ N_5 = -\frac{4}{21}\psi_5 - \frac{8}{45}\psi_7, \ \ N_7 = -\frac{4}{45}\psi_7. \tag{A17}$$

Following the regularity requirement on V'_z and Eqs. (39) the coefficient of $\cos\theta \sim (1-\zeta^2)^{1/2}$ in (A16) must is zero, namely:

$$N_1 = \frac{8}{5}\psi_3 + \frac{32}{21}\psi_5 + \frac{64}{45}\psi_7 = 0. \tag{A18}$$

Applying relations (A12) -(A18) to Eq. (38) for $\hat{n}^{(1)}$ yields:

$$G = G_3 \cos^3\theta + G_5 \cos^5\theta + G_7 \cos^7\theta, \tag{A19}$$

and

$$\psi_j^{(1)} = \mp i\frac{9}{8}\lambda G_j, \quad \lambda = m \pm 1. \tag{A20}$$

Here $\psi_j^{(1)}$ are the coefficient for the right hand side of Eq. (38) for $\hat{n}^{(1)}$, the coefficients G_j will be specified below ($j = 3, 5, 7$). The above relations determine $\hat{n}^{(1)}$

$$\hat{n}^{(1)} = N_1^{(1)} \cos\theta + N_3^{(1)} \cos^3\theta + N_5^{(1)} \cos^5\theta + N_7^{(1)} \cos^7\theta, \tag{A21}$$

where $N_j^{(1)}$ are expressed in terms of $\psi_j^{(1)}$, ($j = 1, 3, 5, 7$), through relations (A17)

$$N_1^{(1)} = \mp i\lambda(\frac{9}{5}G_3 + \frac{12}{7}G_5 + \frac{8}{5}G_7), \quad N_3^{(1)} = \pm i\lambda(\frac{9}{10}G_3 + \frac{6}{7}G_5 + \frac{4}{5}G_7), \quad N_5^{(1)} = \pm i\lambda(\frac{3}{14}G_5 + \frac{1}{5}G_7), \quad N_7^{(1)} = \pm i\lambda\frac{1}{10}G_7, \tag{A22}$$

with the regularity condition

$$N_1^{(1)} = \mp i\lambda(\frac{9}{5}G_3 + \frac{12}{7}G_5 + \frac{8}{5}G_7) = 0, \quad \lambda = m \pm 1. \tag{A23}$$

Hence $\hat{n}^{(1)}$ is determined through G_j ($j = 3, 5, 7$), which will be specified below.

Applying relations (A12) -(A18) to Eq. (40) for $\hat{n}^{(0)}$ yields:

$$\psi_j^{(0)} = \mp\frac{3}{2}N_j^{(1)} - \frac{3}{4}i(m \pm 2)G_j, \quad (j = 1, 3, 5, 7). \tag{A24}$$

Substituting (A22) into (A24) yields

$$\psi_1^{(0)} = -\frac{3}{4}i(m\pm2)G_1, \quad \psi_3^{(0)} = -\frac{3}{20}i(14m\pm19)G_3 - \frac{9}{7}i(m\pm1)G_5 - \frac{6}{5}i(m\pm1)G_7,$$

$$\psi_5^{(0)} = -\frac{3}{28}i(10m \pm 17)G_5 - \frac{3}{10}i(m \pm 1)G_7, \quad \psi_7^{(0)} = -\frac{3}{20}i(6m \pm 11)G_7, \tag{A25}$$

and

$$\hat{n}^{(0)} = N_1^{(0)} \cos\theta + N_3^{(0)} \cos^3\theta + N_5^{(0)} \cos^5\theta + N_7^{(0)} \cos^7\theta. \tag{A26}$$

The coefficients $N_j^{(0)}$ are expressed through $\psi_j^{(0)}$ according to (A17) and then by using (A25) through G_j, ($j = 1, 3, 5, 7$). Once again, requiring the regularity of V_z' and employing Eq. (41) results in:

$$N_1^{(0)} = -i\frac{6}{25}(14m \pm 19)G_3 - i\frac{8}{245}(113m \pm 148)G_5 - i\frac{16}{175}(40m \pm 37)G_7 = 0. \tag{A27}$$

Finally, setting, e.g. $G_3 = 1$ as an arbitrary amplitude of the inertia-coriolis driving waves, Eqs. (A23) and (A27) determine the coefficients G_5 and G_7. Since both $F(r)$ and $G(\zeta)$ are known, the corresponding velocities $\hat{v}_r$, $\hat{v}_\phi$, $\hat{v}_z^{(k)}$ ($k = 0, 1$) may be calculated along with the number density.

REFERENCES

Balbus, S. A., & Hawley, J. F., 1991, ApJ, **376**, 214.
Bodo G., Mignone, A., Cattaneo, F., Rossi, P., & Ferrari, A., 2008, A&A, **487**, 1.
Boberg, L. & Brosa, U., 1988, Z. Naturferschung, **43a**, 697.
Boubnov B.M., Glezder, E.B., & Hopfinger, E.J., J. Fluid Mech., **292**, 333.
Butler, K. M., & Farrell, B. F., Phys. Fluids A, **4**, 1637.
Chun, S.T., & Hameiri, E., 1990, Phys. Fluids B, **2**, 1745.
Coppi, B., & Keyes, E.A., 2003, ApJ, **595**, 1000.
Criminale, W.O., Jackson, T.L., Lasseigne, D.G., & Joslin, R.D., 1997, J. Fluid Mech., **339**, 55.
Dubrulle, B., Marié, L., Normand, C., Richard, D., Hersant, F., & Zahn, J.-P., 1997, A&A, **429**, 1.
Frank J., King A., & Raine D., 2002, *Accretion Power in Astrophysics*, (Cambridge University Press)
Fromang S., & Papaloizou, J., A&A, 2007, **476**, 1113.
Fromang S., Papaloizou, J., Lesur, G., & Heineman, T., 2007, A&A, **476**, 1123.
Goldreich, P., & Lynden-Bell, D., 1965, ApJ, **623**, 1157.
Grossmann, S., 2000, Rev. Modern Phys., **72**, 603.
Gustavsson, L.H., 1991, J. Fluid Mech., **224**, 241.
Kluzniak, W., & Kita, D., 2000, astro-ph/0006266.
Lesur G., & Longaretti P.Y., 2007, MNRAS, **378**, 1471.
Lesur G., & Longaretti P.Y., 2007, A&A, **444**, 25.

Liverts, E., & Mond, M., 2009, MNRAS, **392**, 287.
Ogilvie, G. I., 1997, MNRAS, **288**, 63.
Pessah, M.E., Chan C.K., & Psaltis, D., 2007, ApJ, **668**, L51.
Rebusco, P., Umurhan, O. M., Kluzniak, W., & Regev O., 2009, Phys. Fluids, **21**, 076601.
Reddy, S. C. & Henningson, D. S., 1993, J. Fluid Mech., **252**, 209.
Regev, O., 1983, A&A, **126**, 146.
Regev, O., & Umurhan O.M., 2008, A&A, **481**, 21.
Regev, O., & Gitelman L., 2002, A&A, **396**, 623.
Reynolds, C.S., & Miller M.C., 2009, Ap.J., **692**, 869.
Schmid P.J., & Henningson, D.S., 2000, *Stability and Transition in Shear Flows*, (Springer)
Shalybkov D. & Rüdiger G., 2005, A&A, **438**, 1157.
Schmid, P.J., 2007, Annu. Rev. Fluid Mech., **39**, 129.
Shtemler, Y. M., Mond., M., & Liverts., E., 2007, ApJ, **665**, 1371.
Shtemler Y. M., Mond M., & Rüdiger G., 2009, MNRAS, **394**, 1379.
Tevzadze, A.G., Chagelishvili, G.D., & Zahn, J.-P., 2008, A&A, **478**, 9.
Trefethen, L. N., Trefethen, A. E., Reddy, S. C. & Driscoll, T. A., 1993, Science, **261**, 578.
Umurhan, O. M., 2006, MNRAS, **365**, 85.
Umurhan, O. M., Nemirovvsky, A., Regev, O., & Shaviv, G., 2006, A&A, **446**, 1.
Umurhan, O. M., & Shaviv, G., 2005, A&A, **432**, L31.
Yavneh,I., McWilliams, J.C., and Molemaker, M.J., 2001, J. Fluid Mech. **448**, 1.

This paper has been typeset from a TEX/ LATEX file prepared by the author.