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Current-driven instabilities in weakly ionized disks

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Cool weakly ionized gaseous rotating disk form the basis for many models in astrophysics objects. Instabilities against perturbations in such disks play an important role in the theory of the formation of stars and planets. Traditionally, axisymmetric magnetohydrodynamic (MHD) and recently Hall-MHD instabilities have been thoroughly studied as providers of an efficient mechanism for radial transfer of angular momentum, and of density radial stratification. In the current work, the Hall instability against axisymmetric perturbations in incompressible rotating fluid in external poloidal and toroidal magnetic field is considered.

1 Introduction

The origin of turbulence in astrophysical disks is often attributed to hydrodynamic and hydromagnetic instabilities that can occur in differentially rotating stratified gas. The magnetorotational instability (MRI) is usually considered as one of the possible candidates because it can operate in a conductive flow if the angular velocity decreases with the cylindrical radius (Velikhov 1959). The MRI has been studied in detail for both stellar and accretion disk conditions (see, e.g., Fricke 1969; Safronov 1969; Acheson 1978,1979; Balbus & Hawley 1991; Kaisig, Tajima & Lovelace 1992; Zhang, Diamond & Vishniac 1994). Note that the MRI occurs only in the presence of a weak magnetic field, because a sufficiently strong field can suppress the instability completely. Numerical simulations of the MRI in accretion disks (Hawley, Gammie & Balbus 1995; Brandenburg et al. 1995;Matsumoto & Tajima 1995;Torkelsson et al. 1996;Arlt & Rüdiger 2001) show that the turbulence generated can enhance essentially the angular momentum transport.

The properties of turbulence in low conductive protostellar disks can differ essentially from those of accretion disks. The magnetic Reynolds number is not very large in protostellar disks and, hence, the field cannot be treated as 'frozen' into the gas (Gammie 1996). The influence of ohmic dissipation on the MRI has been considered in the linear (Jin 1996) and nonlinear regimes (Sano Inutsuka & Miyama 1998). Calculations of many authors indicate that the MRI is unlikely to be the source of turbulence in protostellar disks since it arises only in a highly conductive plasma. For instance, Turner, Sano & Dziourkevitch (2006) have considered in detail turbulent mixing caused by the MRI in protostellar disks and argued that the MRI does not arise in the midplane even under the most favorable conditions because the midplane is shielded from cosmic rays which are the main ionizing factor. However, the number of instabilities that occur in disks is not restricted by the MRI alone. An analysis of MHD modes in stratified disks demonstrates a wide variety of instabilities even in the case of a very simple magnetic geometry (Keppens, Casse & Goedbloed 2002). More complex magnetic geometries with a non-vanishing radial field lead to additional instabilities (Bonanno & Urpin 2007, 2008). Generally, even a pure hydrodynamic origin of turbulence cannot be excluded (see, e.g., Dubrulle et al. 2005; Lesur & Longaretti 2005).

It was first pointed out by Wardle (1999) that poorly conducting protostellar disks can be strongly magnetized if electrons are the main charge carriers. As a result, transport must be anisotropic with substantially different properties along and across the magnetic field. The effect of the magnetic field on transport properties of plasma is characterized by the magnetization parameter $a_{\rm e} = \omega_B \tau$ where $\omega_B = eB/m_{\rm e}c$ is the gyrofrequency of electrons and τ is their relaxation time (see, e.g., Spitzer 1978). In protostellar disks, τ is determined by the scattering of electrons on neutrals and can be calculated by making use of the fitting expression for the cross-section obtained by Draine Roberge & Dalgarno (1983). Then, we have for the magnetization parameter

$$a_{\rm e} \approx 21 \, B n_{14}^{-1} T_2^{-1/2}, \tag{1}$$

where B is the magnetic field measured in Gauss, $n_{14} = n/10^{14}$ cm^{-3} and $T_2 = T/100$ K with n and T being the number density of neutrals and temperature, respectively. If $a_{\rm e} > 1$, i.e. $B > 0.048 \; n_{14}\sqrt{T_2}$ G, then the electron transport is anisotropic, and the magnetic diffusivity is represented by a tensor. In a weakly ionized plasma of protostellar disks, the difference between the components of the magnetic diffusivity which are parallel and perpendicular

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to the magnetic field is small (see, e.g., Balbus & Terquem 2001), but the Hall component that is perpendicular to the both magnetic field and electric current can be much greater. The Hall component of diffusivity is given by $a_{\rm e}\eta$ where $\eta = c^2 m_{\rm e}/4\pi e^2 n_{\rm e}\tau$ is the magnetic diffusivity at $B = 0$; $m_{\rm e}$ and $n_{\rm e}$ are the mass and number density of electrons, respectively. Using the fit for the cross-section by Draine Roberge & Dalgarno (1983), we obtain for the magnetic diffusivity

$$\eta = 2.34 \times 10^3 x_e^{-1} T_2^{1/2} \ \ {\rm cm}^2 \ {\rm s}^{-1}, \tag{2}$$

where $x_e = n_{\rm e}/n$ is the ionization fraction.

A stability analysis of the MRI done by Wardle (1999) shows that the Hall effect can provide either stabilizing or destabilizing influence depending on the direction of the field. A more general consideration of the Hall MRI has been done by Balbus & Terquem (2001). They found that the Hall effect changes qualitatively the stability properties of protostellar disks and can lead to instability even if the angular velocity increases outward. These authors, however, did not take into account the effect of gravity that is crucial for disks. A consistent consideration of the linear MRI under the combined influence of the Hall effect and gravity has been done by Urpin & Rüdiger (2005) who derived also the criteria of several other instabilities that can occur in protostellar disks. The properties of the MRI modified by the Hall effect has been considered also by Salmeron & Wardle (2005). These authors argued that the MRI is active in protoplanetary disks over a wide range of field strengths and fluid conditions. The Hall conductivity results in a faster growth of perturbations and extends the region of instability. Recently, Liverts Mond & Chernin (2007) and Shtemler, Mond & Liverts (2007) have considered the Hall instability (HI) for non-axisymmetric perturbations. This instability differs from the MRI and it results from the fast magnetosonic waves in contrast to the Alfven nature of the MRI. The HI instability is proposed as a viable mechanism for the azimuthal fragmentation of the protoplanetary disks and planet formation. The non-axisymmetric instability is caused basically by the combined effect of the radial stratification and Hall electric field.

Apart from the Hall effect, the stability properties of magnetic protostellar disks can be influenced by a number of other factors, for example, the electric currents. In differentially rotating disks, the azimuthal field is generated by stretching from the poloidal one, and poloidal currents are necessary to maintain this azimuthal field that typically is not current-free. The generated azimuthal field can be stronger than the poloidal one if the magnetic Reynolds number is larger than 1. The azimuthal field and associated currents can be important for stability of disks even if the Hall effect is negligible (see Pessah & Psaltis 2005). In protostellar disks, the effect of currens maintaining the magnetic configuration is accompinied often by the Hall effect that changes crucially the stability properties.

In the present paper we consider the linear stability properties of magnetic protostellar disks taking into account the combined influence of the Hall effect and electric currents. The criteria of instability are derived, and the growth rate of the various modes is calculated.

2 Basic equations

Consider the stability of a magnetized protostellar disk of a finite vertical extent. For the sake of simplicity, the unperturbed angular velocity is assumed to be dependent on the cylindrical radius s alone such as $\Omega = \Omega(s)$; (s, φ, z) are cylindrical coordinates. The magnetic field, $\boldsymbol{B} = (B_s, B_\varphi, B_z)$, is assumed to be weak in the sense that the Alfvén speed, $c_{\rm A}$, is small compared to the sound speed, $c_{\rm s}$. This enables us to employ the Boussinesq approximation for a consideration of slowly varying modes. In the unperturbed state, the disk is assumed to be in hydrostatic equilibrium in the s- and z-directions,

$$\frac{\nabla p}{\rho} = \boldsymbol{G} + \frac{1}{4\pi\rho} {\rm rot} \boldsymbol{B} \times \boldsymbol{B} \ \ , \quad \boldsymbol{G} = \boldsymbol{g} + \Omega^2 \boldsymbol{s} \ , \tag{3}$$

here $\boldsymbol{g}$ is the gravity force per unit mass. However it should be noticed that the unperturbed Lorentz force namely the second term on the r.h.s. Eq. (3) is usually much smaller than gravity and centrifugal forces (the first term on the r.h.s. of the last equation), thus the pressure in the unperturbed disks is mainly determined by gravity and centrifugal forces.

We consider the stability of axisymmetric short wavelength perturbations with space-time dependence $\exp(\gamma t - i\boldsymbol{k}\cdot\boldsymbol{x})$ where $\boldsymbol{k} = (k_s, 0, k_z)$ is the wave vector. The linearized momentum and continuity equations read in the Boussinesq approximation

$$\gamma \boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{e}_\varphi s (\boldsymbol{v}\cdot\nabla)\Omega = \frac{i\boldsymbol{k}p}{\rho} - \alpha \boldsymbol{G} T_1 +$$

$$\frac{i}{4\pi\rho}[(\boldsymbol{B}\cdot\boldsymbol{b})\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{b}] - \frac{\boldsymbol{e}_r}{4\pi\rho}\frac{B_\varphi b_\varphi}{s} + \frac{1}{c\rho}\boldsymbol{J}\times\boldsymbol{b}, \tag{4}$$

$$\boldsymbol{k}\cdot\boldsymbol{v} = 0, \tag{5}$$

where $\boldsymbol{v}$, $\boldsymbol{b}$, p and T_1 are the perturbations of the hydrodynamic velocity, magnetic field, pressure and temperature, respectively; $\alpha = -(\partial \ln\rho/\partial T)_P$ is the thermal expansion coefficient and $\boldsymbol{e}_\varphi$ is the unit vector in the azimuthal direction. It is assumed in Eq. (4) that the density perturbation in the buoyancy force is determined by the temperature perturbation alone in accordance with the main idea of the Boussinesq approximation, $\rho_1 = -\rho\alpha T_1$, and the unperturbed Lorentz force in Eq. (3)is neglected. We took into account the effect of electric currents (the last term on the r.h.s. of Eq. (4)). In short wavelength approximation, this term seems to be smaller than the previous one by a factor $\sim kL \gg 1$, where L ($\sim s$) is the length-scale of the unperturbed magnetic field. In differentially rotating discs, however, the electric current $\boldsymbol{J} = (c/4\pi)\nabla\times\boldsymbol{B}$ is mainly determined by the ϕ-component of the magnetic field that can be substantially stronger than B_s and B_z because of stretching the magnetic field lines in the azimuthal direction by differential rotation. If B_φ satisfies the condition

$$B_\varphi > kL \max(B_s, B_z), \tag{6}$$

then the effect of electric currents cannot be neglected in Eq. (4). For the sake of simplicity, we assume that the toroidal field B_φ depends on s alone, then

$$\boldsymbol{J} = J_z \boldsymbol{e}_z \ , \ \ J_z = \frac{c}{4\pi s}\frac{\partial}{\partial s}(s B_\varphi). \tag{7}$$

Note that, calculating the perturbation of the electric current $\boldsymbol{j} = (c/4\pi)\nabla \times \boldsymbol{b}$ in Eq. (4), we can neglect terms of the order of $1/s$ and assume $\boldsymbol{j} \approx -(ic/4\pi)\boldsymbol{k}\times\boldsymbol{b}$. Using Eq. (5), we can calculate p from Eq. (4). Then, we obtain for the momentum equation

$$\gamma \boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v} - \frac{2\boldsymbol{k}}{k^2}\boldsymbol{k}\cdot(\boldsymbol{\Omega}\times\boldsymbol{v}) + \boldsymbol{e}_\varphi s\Omega' v_s =$$
$$-\alpha T_1 \left[\boldsymbol{G} - \frac{\boldsymbol{k}}{k^2}(\boldsymbol{k}\cdot\boldsymbol{G})\right] - \frac{i}{4\pi\rho}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{b}$$
$$+\frac{1}{c\rho}\left[\boldsymbol{J}\times\boldsymbol{b} - \frac{\boldsymbol{k}}{k^2}\boldsymbol{k}\cdot(\boldsymbol{J}\times\boldsymbol{b})\right]. \tag{8}$$

It is clearly seen from the last equation that the third term on the r.h.s. which results from the unperturbed electric current can be larger under condition (6) than the second term which is usually taken into account in a stability analysis of disks and which contains only poloidal components B_s and B_z of the background field.

Since the thermal conductivity of protostellar disks is low because of a low temperature ($T \sim 10 - 10^3$ K), we adopt the adiabatic equation to describe the evolution of temperature perturbations,

$$\gamma T_1 + \boldsymbol{v}\cdot(\Delta\nabla T) = 0\ , \tag{9}$$

where $(\Delta\nabla T) = \nabla T - \nabla_{\rm ad} T$ is the difference between the actual and adiabatic temperature gradients. Substituting T_1 into Eq. (8), we obtain the equation that contains only perturbations of $\boldsymbol{v}$ and $\boldsymbol{b}$. The s- and φ-components of this equation read

$$(\gamma + \frac{\omega_g^2}{\gamma})v_s - 2\mu\Omega v_\varphi = -\frac{i}{4\pi\rho}(\boldsymbol{k}\cdot\boldsymbol{B})b_s - \frac{\mu J_z}{c\rho}b_\varphi, \tag{10}$$

$$\gamma v_\varphi + (2\Omega + s\Omega')v_s = -\frac{i}{4\pi\rho}(\boldsymbol{k}\cdot\boldsymbol{B})b_\varphi + \frac{J_z}{c\rho}b_s, \tag{11}$$

where

$$\omega_g^2 = -\alpha\Delta\nabla T\cdot\left[\boldsymbol{G} - \frac{\boldsymbol{k}}{k^2}(\boldsymbol{k}\cdot\boldsymbol{G})\right]$$

and $\mu = k_z^2/k^2$.

As it was mentioned, the effect of the magnetic field on kinetic properties of plasma is usually characterized by the magnetization parameter $a_{\rm e}$ (see Eq. (1)). We consider the most interesting case for protoplanetary disks when this parameter is moderate, $kL \gg a_e$. Under this assumption, the linearized induction equation reads

$$\gamma\boldsymbol{b} \ = \ -\eta\nabla\times(\nabla\times\boldsymbol{b}) + \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) + \nabla\times(s\Omega\boldsymbol{e}_\varphi\times\boldsymbol{b})$$
$$- \ \frac{c}{4\pi e n_e}\nabla\times[(\nabla\times\boldsymbol{b})\times\boldsymbol{B} + (\nabla\times\boldsymbol{B})\times\boldsymbol{b}], \tag{12}$$

where η is magnetic diffusivity and n_e is the number density of electrons. For short wavelength perturbations, Eq. (12) and the divergence-free condition read

$$(\gamma + i\omega_\wedge)\boldsymbol{b} = -i\boldsymbol{v}(\boldsymbol{k}\cdot\boldsymbol{B}) - \boldsymbol{e}_\varphi v_s s\frac{\partial}{\partial s}\left(\frac{B_\varphi}{s}\right) + \boldsymbol{e}_\varphi s\Omega' b_s$$
$$-\boldsymbol{e}_z\frac{J_z'}{e n_e}b_s + \frac{c(\boldsymbol{k}\cdot\boldsymbol{B})}{4\pi e n_e}\boldsymbol{k}\times\boldsymbol{b} + \boldsymbol{e}_\varphi\frac{ick_z}{4\pi e n_e}B_\varphi' b_\varphi, \tag{13}$$
$$\boldsymbol{k}\cdot\boldsymbol{b} = 0, \tag{14}$$

where $\omega_\wedge = k_z J_z/e n_e - i\eta k^2$, $B_\varphi' = dB_\varphi/ds$, and $J_z' = dJ_z/ds$; the last three terms on the r.h.s. of Eq. (13) together with the first term in $\omega_\wedge$ represent the Hall effect that can be important in protoplanetary disks. Under condition (6), induction equation (13) can be simplified because the fourth term on the r.h.s is small compared to the last term and do not influence the behavior of perturbations. Then, we have for the induction equation

$$(\gamma + i\omega_\wedge)\boldsymbol{b} = -i\boldsymbol{v}(\boldsymbol{k}\cdot\boldsymbol{B}) -$$
$$\boldsymbol{e}_\varphi\left[v_s s\frac{\partial}{\partial s}\left(\frac{B_\varphi}{s}\right) - s\Omega' b_s - i\omega_H b_\varphi\right] + \frac{c(\boldsymbol{k}\cdot\boldsymbol{B})}{4\pi e n_e}\boldsymbol{k}\times\boldsymbol{b}, \tag{15}$$

where $\omega_H = ck_z B_\varphi'/4\pi e n_e$. The s- and φ-components of this equation are

$$(\gamma + i\omega_\wedge)b_s = -i(\boldsymbol{k}\cdot\boldsymbol{B})v_s + \omega_w b_\varphi, \tag{16}$$

$$(\gamma + i\omega_0)b_\varphi = -i(\boldsymbol{k}\cdot\boldsymbol{B})v_\varphi - s\frac{\partial}{\partial s}\left(\frac{B_\varphi}{s}\right)v_s -$$
$$\frac{i\boldsymbol{k}\cdot\boldsymbol{B}}{\gamma + i\omega_\wedge}s\Omega' v_s - \frac{k^2}{k_z^2}\omega_w b_s\ , \tag{17}$$

where

$$\omega_0 = \omega_\wedge - \omega_H = \frac{ck_z B_\varphi}{2\pi e n_e s} - i\eta k^2, \omega_w = \frac{ck_z(\boldsymbol{k}\cdot\boldsymbol{B})}{4\pi e n_e}. \tag{18}$$

Eqs. (10), (11), (16), and (17) describe the eigenmodes that exist in a magnetized fluid in the presence of electric currents.

3 Dispersion equation and stability criteria

The general dispersion equation for the set of Eqs. (10), (11), (16), and (17) is rather cumbersome. Therefore, we consider only a particular case of perturbations with the wavevector perpendicular to the unperturbed magnetic field, $\boldsymbol{k}\cdot\boldsymbol{B} = 0$. The MRI does not occur for such perturbations, and they should be stable if the Hall effect is neglected and $\boldsymbol{J} = 0$. Therefore, the particular case $\boldsymbol{k}\cdot\boldsymbol{B} = 0$ allows to study a destabilizing influence of both the unperturbed electric current and the Hall effect in a situation when other factors can cause only a stabilizing influence. If $\boldsymbol{k}\cdot\boldsymbol{B} = 0$, then the dispersion equation takes the form

$$\gamma^3 + a_2\gamma^2 + a_1\gamma + a_0 = 0, \tag{19}$$

where

$$a_2 = i\omega_0, a_1 = \omega_g^2 + \mu(\kappa^2 + \omega_J^2), a_0 = i\omega_0(\omega_g^2 + \mu\kappa^2). \tag{20}$$

The characteristic frequencies in these expressions are given by

$$\omega_0 = \omega_1 - i\omega_\eta, \;\; \omega_1 = \frac{ck_z B_\varphi}{2\pi e n_e s}, \;\; \omega_\eta = \eta k^2,$$
$$\kappa^2 = 2\Omega(2\Omega + s\Omega'), \;\; \omega_J^2 = -\frac{sJ_z}{c\rho}\frac{\partial}{\partial s}\left(\frac{B_\varphi}{s}\right) \qquad (21)$$

For $\boldsymbol{k}\cdot\boldsymbol{B} = 0$, only three non-trivial modes exist in the flow but other modes are degenerate.

3.1 Stability in the case $a_e \ll 1$

If the Hall parameter is small, the Hall effect does not influence the stability properties. In this case, $\omega_0 \approx -i\omega_\eta$. Then, the dispersion equation takes the form

$$\gamma^3 + b_2\gamma^2 + b_1\gamma + b_0 = 0, \qquad (22)$$

where all coefficients of this equation are real,

$$b_2 = \omega_\eta, \quad b_1 = a_1 = \omega_g^2 + \mu(\kappa^2 + \omega_J^2),$$
$$b_0 = \omega_\eta(\omega_g^2 + \mu\kappa^2). \qquad (23)$$

The condition that at least one of the roots of Eq. (22) has a positive real part (that corresponds to instability) is equivalent to one of the following inequalities

$$b_2 < 0, \quad b_1 b_2 < b_0, \quad b_0 < 0 \qquad (24)$$

being fulfilled (see, e.g., Aleksandrov, Kolmogorov & Laurentiev 1985). Since $\omega_\eta > 0$, the first condition $b_2 < 0$ will never apply. The other two conditions yield

$$\mu\omega_\eta\omega_J^2 < 0 \;\;, \quad \omega_\eta(\omega_g^2 + \mu\kappa^2) < 0. \qquad (25)$$

Both criteria are proportional to the dissipative frequency (that is positively defined quantity) and appear only if one takes into account magnetic diffusivity. Therefore, both criteria describe diffusive instabilities that can be relatively fast in protostellar disks. In ideal magnetohydrodynamics, we have $\omega_\eta = 0$ and $b_0 = b_2 = 0$, and dispersion relation (19) transforms into

$$\gamma^2 + a_1 = 0. \qquad (26)$$

The condition of instability reads $a_1 < 0$, or

$$\omega_g^2 + \mu(\kappa^2 + \omega_J^2) < 0, \qquad (27)$$

that differ substantially from dissipative criteria (25). For instance, to satisfy condition (27) in disks, energy of the magnetic field should be comparable to the rotational or gravitational energy of gas. The first condition (25) can be fulfilled even in a much weaker field. Since $\omega_\eta > 0$ and $\mu > 0$, we obtain from Eq. (25) the following conditions of instability

$$\omega_J^2 < 0, \qquad (28)$$
$$\omega_g^2 + \mu\kappa^2 < 0. \qquad (29)$$

Eq. (29) is the standard criterion of convection modified by rotation and usually is not satisfied in astrophysical discs. Eq. (28) is the condition of an instability that can occur due to the presence of electric currents (see also Velikhov 1959). This instability is associated only with the distribution of electric currents in disks. Condition (28) can be rewritten as

$$B_\varphi^2 - s^2 B_\varphi'^2 < 0. \qquad (30)$$

Therefore, the current-driven instability arises if B_φ decreases with s faster than $1/s$ or increases outward faster than s. Note also that the current-driven instability does not depend on the angular velocity profile and can occur for perturbations that are not subject to the MRI since $\boldsymbol{k} \perp \boldsymbol{B}$.

Since the coefficients of equation (22) are real there exist three real roots or one real and two complex conjugate roots. The number of roots with a positive real part is determined by Routh criterium (DiStefano III, Stubberud & Williams 1994), which states that the number of unstable modes of a cubic equation (22) is given by the number of changes of sign in the sequence

$$\left\{1, \; b_2, \; \frac{b_2 b_1 - b_0}{b_2}, \; b_0\right\}. \qquad (31)$$

For coefficients (23), this sequence reads

$$\{1 \;, \;\; \omega_\eta \;, \;\; \omega_J^2 \;, \;\; \omega_\eta(\omega_g^2 + \mu\kappa^2)\}. \qquad (32)$$

If the disc is convectively stable ($\omega_g^2 + \mu\kappa^2 > 0$), we obtain that under the condition (28) (or (30)) two complex conjugate modes are unstable. If the disc is convectively unstable ($\omega_g^2 + \mu\kappa^2 < 0$) and the condition (28) holds then there should be only one unstable mode. Only one mode is unstable also in the case when the disc is convectively unstable but the condition (28) is not fulfilled.

The roots γ_i ($i = 1, 2, 3$) of the cubic equation (19) can be represented as $\gamma_i = x_i - a_2/3$. The expressions for x_i are

$$x_1 = u + v \;, \;\; x_{2,3} = -\frac{1}{2}(u+v) \pm i\frac{\sqrt{3}}{2}(u-v) \;, \qquad (33)$$

where

$$(u, v) = (-q \pm \sqrt{q^2 + p^3})^{1/3} \;,$$
$$2q = \frac{2}{27}a_2^3 - \frac{1}{3}a_2 a_1 + a_0 \;, \;\; 3p = a_1 - \frac{1}{3}a_2^2 \;. \qquad (34)$$

(see, e.g., Bronstein & Semendyaev 1957). In a particular case $a_{\rm e} < 1$ when the coefficients of a cubic equation are given by Eq. (23), we have

$$q = \frac{\omega_\eta}{3}\left(\omega_g^2 + \mu\kappa^2 - \frac{1}{2}\mu\omega_J^2 + \frac{1}{9}\omega_\eta^2\right),$$
$$p = \frac{1}{3}\left(\omega_g^2 + \mu\kappa^2 + \mu\omega_J^2 - \frac{1}{3}\omega_\eta^2\right).$$

In the limit of small ω_η, we have for the roots

$$\gamma_1 = -\frac{\omega_\eta(\omega_g^2 + \mu\kappa^2)}{\omega_g^2 + \mu\kappa^2 + \mu\omega_J^2}, \qquad (35)$$

$$\gamma_{2,3} = \pm i\sqrt{\omega_g^2 + \mu\kappa^2 + \mu\omega_J^2} - \frac{1}{2}\frac{\omega_\eta\mu\omega_J^2}{\omega_g^2 + \mu\kappa^2 + \mu\omega_J^2}. \qquad (36)$$

In this case, the instability is dissipative since the growth rate is proportional to η.

If $\omega_g^2 + \mu(\kappa^2 + \omega_J^2) > 0$, then there should be no instability in the ideal magnetohydrodynamics (see condition (27)). Indeed, expressions (36) and (37) yield ${\rm Re}\gamma = 0$ in the limit $\omega_\eta \to 0$. However, in a dissipative MHD, the instability can occur even if $\omega_g^2 + \mu(\kappa^2 + \omega_J^2) > 0$. In this case, the first mode (non-oscillatory) is unstable alone if $\omega_g^2+\mu\kappa^2<0$, but two oscillatory modes are stable since ω_J^2 should be positive. On the contrary, if $\omega_g^2+\mu(\kappa^2+\omega_J^2)>0$ but $\omega_J^2<0$, then two oscillatory modes are unstable, but the first mode should be stable since $\omega_g^2+\mu\kappa^2$ is positive. If $\omega_g^2+\mu(\kappa^2+\omega_J^2)<0$, then mode 3 is unstable with a very large growth rate $\gamma_3=\sqrt{|\omega_g^2+\mu(\kappa^2+\omega_J^2)|}$ but mode 2 is rapidly decaying with the decay rate approximately equal to γ_3. The first mode can also be unstable in this case if $\omega_g^2+\mu\kappa^2>0$ but its growth rate is small since it is proportional to the small dissipative frequency ω_η.

If ω_η is greater than other characteristic frequencies, then we have from Eqs. (33)-(35)

$$\gamma_1\approx-\omega_\eta,\quad \gamma_{2,3}\approx\pm i\sqrt{\omega_g^2+\mu\kappa^2}-\frac{\mu\omega_J^2}{2\omega_\eta}. \qquad (37)$$

Mode 1 is always stable but modes 2 and 3 can be unstable. If $\omega_g^2+\mu\kappa^2>0$, then modes 2 and 3 are oscillatory. The instability of these modes occurs if $\omega_J^2<0$. In the opposite case $\omega_J^2>0$, oscillatory modes do not arise. If $\omega_g^2+\mu\kappa^2<0$, then oscillatory modes become non-oscillatory, and one of these modes is unstable. The case of large ω_η corresponds to small magnetic Reynolds number and is of particular interest for protostellar disks. The instability of oscillatory modes can operate even in the dead zone of protoplanetary disks where the magnetic Reynolds number is small.

3.2 Stability in a strong magnetic field with $a_e\geq 1$

Let us consider the stability of a strongly magnetized plasma with $a_e\geq 1$. In this case, the growth rate is descibed by Eq. (19) with complex coefficients. The roots of Eq. (19) can be calculated by making use of general expressions (33)-(34) for the roots of a cubic equation. However, these expressions are rather cumbersome and inconvenient for analysis. Therefore, we consider in detail the growth rate in the case when the frequency associated to electric currents ω_J is lower than the angular velocity Ω or characteristic buoyancy frequency ω_g. The dissipative frequency ω_η can be high and comparable to (or even higher than) other characteristic frequencies. This case is of particular interest for the dead zones of protostellar disks where the conductivity is extremely low, and the magnetic Reynolds number can be relatively small. We can rewrite Eq. (19) as

$$(\gamma^2+\omega_g^2+\mu\kappa^2)+\frac{\gamma\mu\omega_J^2}{\gamma+i\omega_0}=0. \qquad (38)$$

This shape is more convenient to calculate the oscillatory roots (which can generally be unstable) by making use of a perturbation procedure. Since the last term on the l.h.s. is proportional to the square of a low frequency ω_J, it can be considered as a small perturbation. Therefore, the solution of Eq. (38) can be represented as a power series of ω_J^2: $\gamma=\gamma^{(0)}+\gamma^{(1)}+...$ where $\gamma^{(0)}$ does not depend on ω_J^2 and $\gamma^{(1)}$ is linear in ω_J^2. The equation of the zeroth order yields

$$\gamma^{(0)}=\pm i\sqrt{\omega_g^2+\mu\kappa^2} \qquad (39)$$

(we assume that the unperturbed disk is convectively stable and $\omega_g^2+\mu\kappa^2>0$). The roots are imaginary in the zeroth approximation, and there is no instability in the absence of electric currents. The correction of the first order is

$$\gamma^{(1)}=-\frac{1}{2}\frac{\mu\omega_J^2}{\gamma^{(0)}+i\omega_0}. \qquad (40)$$

Splitting this equation into real and imaginary parts, we have for the growth rate

$$\mathrm{Re}\gamma=-\frac{1}{2}\frac{\omega_\eta\mu\omega_J^2}{(\omega_1\pm\sqrt{\omega_g^2+\mu\kappa^2})^2+\omega_\eta^2}. \qquad (41)$$

Like the case of a weakly magnetized disk, the instability arises only if the magnetic field satisfies condition (28). The growth rate depends on the wavelength of perturbations and can be essentially different for different k. If the wavelength $\lambda=2\pi/k$ is sufficiently short such as $\omega_\eta>\Omega$, then the growth rate is approximately given by

$$\mathrm{Re}\gamma\approx-\frac{\mu\omega_J^2}{\omega_\eta}. \qquad (42)$$

The order of magnitude estimate of Reγ is

$$\mathrm{Re}\gamma\sim 4.3\times10^{-5}\frac{B_{\varphi2}^2 x_{e-12}}{n_{14}T_2^{1/2}}\left(\frac{\lambda}{s}\right)^2\ \mathrm{s}^{-1}, \qquad (43)$$

where $B_{\varphi2}=B_\varphi/100$G and $x_{e-12}=x_e/10^{-12}$. The condition $\omega_\eta>\Omega$ is equivalent to

$$\lambda_{12}>0.66T_2^{1/2}P_{yr}x_{e-12}^{-1}, \qquad (44)$$

where P_{yr} is the rotation period in years and $\lambda_{12}=\lambda/10^{12}$ cm.

The imaginary part of γ that determines the frequency of perturbations is approximately given by expression of the zerouth order (40). This expression describes buoyancy waves modified by differential rotation. It is well known that the buoyancy waves can be unstable and are responsible for convection if $\omega_g^2<0$ that is not likely to be satisfied in protostellar disks. The instability considered in this section is the instability of buoyant waves as well and, in fact, is an oscillatory modification of convection. In stellar hydrodynamics, an oscillatory convection is often called semiconvection and can be caused, for example, by a gradient of the chemical composition, As it is seen from our consideration, the distribution of the toroidal field can also be the reason of semiconvection in protostellar disks.

In the opposite case, $\omega_\eta<<\Omega$, we have from Eq. (42) for the growth rate

$$\mathrm{Re}\gamma\approx-\frac{1}{2}\ \frac{\omega_\eta\mu\omega_J^2}{\omega_g^2+\mu\kappa^2}. \qquad (45)$$

The instability can occur for perturbations with $\omega_\eta<\Omega$ as well, however, the growth rate is low in this case. The frequency of such weakly unstable perturbations is given by Eq. (40).

4 Summary and discussion

This paper examines the instability of weakly ionized protostellar disks threaded by an external magnetic field. The conductivity of such disks is low, and dissipative effects can play an important role in the evolution of perturbations. This concerns particularly the midplane that is shielded from cosmic rays to such extent that the MRI does not occur even under the most favorable conditions (Turner et al. 2006). Since the problem of stability is rather cumbersome with taking account of dissipative effects, we have considered a special case of perturbations with the wavevector perpendicular to the magnetic field. Such perturbations are not subject to the magnetorotational instability, because its growth rate is proportional to $(\boldsymbol{k} \cdot \boldsymbol{B})$ and is vanishing for the considered perturbations.

It turns out that dissipative process can alter drastically the stability properties of protostellar disks. Apart from the instabilities that are typical for non-dissipative differentially rotating disks, new instabilities can occur that are determined by dissipative processes. In the simplest case considered in this paper, condition of the dissipative instability (30) is entirely determined by the radial dependence of the azimuthal field and can be represented as

$$\left| \frac{d \ln B_{\varphi}}{d \ln s} \right| > 1. \tag{46}$$

The instability arises if the magnetic field decreases with s faster than $1/s$ or increases more rapidly than s. Note that the condition of the considered instability does not depend on the magnetic diffusivity, and can be satisfied in the limiting cases of very high ($\eta \to 0$) and very low conductivity ($\eta \to \infty$). However, the growth rate of instability depends sensitively on conductivity and is proportional to η and $1/\eta$ in high- and low-conductivity limits, respectively.

The condition of instability (37) does not depend directly on the rotation law. However, differential rotation can influence this condition indirectly because the radial profile of B_{φ} depends on $\Omega(s)$. If stretching of the azimuthal field lines in the basic state is balanced by ohmic dissipation, then we approximately have from the induction equation

$$\eta \Delta B_{\varphi} \sim s \Omega' B_s, \tag{47}$$

or

$$B_{\varphi} \sim \frac{s^3 \Omega' B_s}{\eta}. \tag{48}$$

If rotation is Keplerian and $\Omega \propto s^{-3/2}$, then the radial dependence of B_{φ} is given by $B_{\varphi} \propto s^{1/2} B_s / \eta$. Therefore, the considered instability may occur in the disk if the ratio B_s/η decreases with s faster than $s^{-3/2}$. Likely, this condition can often be fulfilled in protostellar disks, particularly if the radial field component decreases as a dipole ($\propto s^{-3}$). Therefore, the considered instability can generate turbulence in regions with a very low conductivity including the dead zones which likely exist in protostellar disks.

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